

Exact relations between M2-brane theories with and without Orientifolds

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Abstract

We study partition functions of low-energy effective theories of M2-branes, whose type IIB brane constructions include orientifolds. We mainly focus on circular quiver superconformal Chern-Simons theory on S^3 , whose gauge group is $O(2N+1) \times USp(2N) \times \cdots \times O(2N+1) \times USp(2N)$. This theory is the natural generalization of the $\mathcal{N} = 5$ ABJM theory with the gauge group $O(2N+1)_{2k} \times USp(2N)_{-k}$. We find that the partition function of this type of theory has a simple relation to the one of the M2-brane theory without the orientifolds, whose gauge group is $U(N) \times \cdots \times U(N)$. By using this relation, we determine an exact form of the grand partition function of the $O(2N+1)_2 \times USp(2N)_{-1}$ ABJM theory, where its supersymmetry is expected to be enhanced to $\mathcal{N} = 6$. As another interesting application, we discuss that our result gives a natural physical interpretation of a relation between the grand partition functions of the $U(N+1)_4 \times U(N)_{-4}$ ABJ theory and $U(N)_2 \times U(N)_{-2}$ ABJM theory, recently conjectured by Grassi-Hatsuda-Mariño. We also argue that partition functions of \hat{A}_3 quiver theories have representations in terms of an ideal Fermi gas systems associated with \hat{D} -type quiver theories and this leads an interesting relation between certain $U(N)$ and $USp(2N)$ supersymmetric gauge theories.

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1 Introduction

In a couple of years, there is remarkable progress in understanding non-perturbative effects in M-theory through gauge/gravity duality. Most important tools in this progress are the supersymmetry localization [1, 2] and Fermi gas approach [3]. These are applied to partition functions in a class of low-energy effective theories of N M2-branes on S^3 and it has turned out that the partition functions are described by an ideal Fermi gas system:

$$Z(N) = \sum_{\sigma \in S_N} (-1)^\sigma \int d^N x \prod_{j=1}^N \rho(x_j, x_{\sigma(j)}), \quad (1.1)$$

where ρ plays an role of the density matrix in the Fermi gas system. Thanks to these techniques, now we know detailed structures of the non-perturbative effects in M-theory on $AdS_4 \times S^7/\mathbb{Z}_k$ [4, 5, 6, 7], which is dual to the 3d $\mathcal{N} = 6$ superconformal Chern-Simons (CS) theory known as the ABJ(M) theory [8, 9] via AdS/CFT correspondence (see also important earlier works [10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20]).

On the other hand, we still do not have detailed understanding of the non-perturbative effects “beyond ABJ(M) theory”, namely more general M2-brane theories with less super-

symmetry¹ (SUSY). For instance, it is unclear whether many attractive features found in the ABJ(M) theory such as the Airy functional behavior [13, 3], pole cancellation [17, 4] and correspondence to topological string [10, 16, 4] are universal for general M2-brane theories or accidental for the ABJ(M) theory. While the Airy functional behavior has been found for a broad class of M2-brane theories [3, 25, 26, 27, 28] and seems universal [29] (see also [30]), the other features have been found in few examples. This problem has been addressed in special cases of Imamura-Kimura type theory [31], whose type IIB brane construction consists of NS5-branes and $(1, k)$ -5 branes connected by N D3-branes. Especially the orbifold ABJM theory and (p, q) model [32] have been studied well in [26, 33, 34, 22, 23, 24]. Also \hat{D} -type quiver theories [27, 28] and O or USp gauge theories with single node [26, 35] have been studied (see also [36]). In order to understand the non-perturbative effects in more detail, it is very important to investigate the non-perturbative effects in various theories of M2-branes.

In this paper we consider a generalization along a different direction. We study partition functions of low-energy effective theories of M2-branes on S^3 , whose type IIB brane constructions include orientifolds. We mainly focus on 3d superconformal CS theory of circular quiver type with the gauge group² $O(2N+1) \times USp(2N) \times \cdots \times O(2N+1) \times USp(2N)$. This theory is a natural generalization of the $O(2N+1)_{2k} \times USp(2N)_{-k}$ ABJM theory with $\mathcal{N} = 5$ SUSY [38, 9]. We show that the S^3 partition function of this type of theory is also described by an ideal Fermi gas system as in (1.1) and its density matrix $\rho_{O(2N+1) \times USp(2N)}$ takes the following form

$$\rho_{O(2N+1) \times USp(2N)}(x, y) = \rho_{U(N)}^{(-)}(x, y), \quad (1.2)$$

where

$$\rho^{(\pm)}(x, y) = \frac{\rho(x, y) \pm \rho(x, -y)}{2}. \quad (1.3)$$

Here $\rho_{U(N)}$ is the density matrix associated with the M2-brane theories without the orientifolds, which are obtained by the replacement $O(2N+1), USp(2N) \rightarrow U(N)$ in the orientifold theories. This indicates that the density matrix for the orientifold theory is the projection of the one without the orientifolds. Introducing the grand canonical partition function by

$$\Xi[\mu] = \sum_N Z(N) e^{\mu N} = \text{Det}(1 + e^\mu \rho), \quad (1.4)$$

the relation (1.2) indicates that the grand partition function of the orientifold theory is related to the one of the non-orientifold theory by

¹ Only exceptions so far are the orbifold ABJM theory and $(2, 2)$ model analyzed in [21] and [22, 23, 24], respectively. The grand potential for the orbifold ABJM theory has a simple relation to the one of the ABJM [21] and the $(2, 2)$ model is expected to be described by topological string on local D_5 del Pezzo [23, 24].

² Recently there appeared a paper [37] on arXiv considering a similar physical setup. This reference mainly considers CS theories of $O(2N) \times USp(2N)$ type, which differs from our setup of $O(2N+1) \times USp(2N)$ type. But we also give some comments on the $O(2N) \times USp(2N)$ type in sec. 3.5.

$$\Xi_{O(2N+1) \times USp(2N)}[\mu] = \Xi_{U(N)}^{(-)}[\mu], \quad (1.5)$$

where $\Xi^{(\pm)}[\mu]$ denotes the grand canonical partition function defined by $\rho^{(\pm)}$. This relation implies that we can obtain non-perturbative information on the orientifold theory from the non-orientifold theory.

Here we present two interesting applications of our main result (1.5). One of them is to determine an exact form of the grand partition function of the $O(2N+1)_{2k} \times USp(2N)_{-k}$ ABJM theory with $k=1$, whose SUSY is expected to be enhanced to $\mathcal{N}=6$ from $\mathcal{N}=5$. This is achieved by combining our result (1.5) with recent results of [39, 40] and we obtain

$$\Xi_{O(2N+1)_2 \times USp(2N)_{-1}}(\mu) = \Xi_{U(N)_1 \times U(N)_{-1}}(\mu/2 + \pi i/2) \cdot \Xi_{U(N)_1 \times U(N)_{-1}}(\mu/2 - \pi i/2). \quad (1.6)$$

Here $\Xi_{U(N)_1 \times U(N)_{-1}}$ is the grand partition function of the $U(N)_1 \times U(N)_{-1}$ ABJM theory, whose exact form is conjectured as [40]

$$\begin{aligned} & \Xi_{U(N)_1 \times U(N)_{-1}}(\mu) \\ &= \exp \left[\frac{3\mu}{4} - \frac{3}{4} \log 2 + \mathcal{F}_1 + F_1^{\text{NS}} - \frac{1}{4\pi^2} \left(\mathcal{F}_0(\lambda) - \lambda \partial_\lambda \mathcal{F}_0(\lambda) + \frac{\lambda^2}{2} \partial_\lambda^2 \mathcal{F}_0(\lambda) \right) \right] \\ & \times \left(\vartheta_2(\bar{\xi}/4, \bar{\tau}/4) + i\vartheta_1(\bar{\xi}/4, \bar{\tau}/4) \right), \end{aligned} \quad (1.7)$$

where several definitions will be given in sec. 2.2.

The other application of (1.5) is to give a natural physical interpretation of a mysterious relation recently conjectured by Grassi-Hatsuda-Mariño [39]. They conjectured a relation between the grand partition functions of the $U(N+1)_4 \times U(N)_{-4}$ ABJ theory and $U(N)_2 \times U(N)_{-2}$ ABJM theory as

$$\Xi_{U(N)_4 \times U(N+1)_{-4} \text{ABJ}}[\mu] = \Xi_{U(N)_2 \times U(N)_{-2} \text{ABJM}}^{(-)}[\mu]. \quad (1.8)$$

This should be compared with our result (1.5) for the $\mathcal{N}=5$ ABJM theory:

$$\Xi_{O(2N+1)_{2k} \times USp(2N)_{-k}}[\mu] = \Xi_{U(N)_{2k} \times U(N)_{-2k}}^{(-)}[\mu]. \quad (1.9)$$

Combining (1.8) with (1.9), we find

$$\Xi_{U(N)_4 \times U(N+1)_{-4} \text{ABJ}}[\mu] = \Xi_{O(2N+1)_2 \times USp(2N)_{-1}}[\mu]. \quad (1.10)$$

Remarkably this relation is indeed equivalent to the conjecture in [9]. The $\mathcal{N}=5$ ABJM theory is expected to be low energy effective theories of N M2-branes probing \mathbb{C}^4/\hat{D}_k with the binary dihedral group \hat{D}_k defined in (2.26). Since \mathbb{C}^4/\hat{D}_k for $k=1$ is $\mathbb{C}^4/\mathbb{Z}_4$, moduli of the $O(2N+1)_2 \times USp(2N)_{-1}$ ABJM theory become the same as the one of the $U(N+M)_k \times U(N)_{-k}$ ABJ(M) with $k=4$. Therefore the work [9] conjectured that the $O(2N+$

$1)_2 \times USp(2N)_{-1}$ ABJM theory has the enhanced $\mathcal{N} = 6$ SUSY and equivalent to the $U(N+1)_4 \times U(N)_{-4}$ ABJ theory³:

$$O(2N+1)_2 \times USp(2N)_{-1} \leftrightarrow U(N+1)_4 \times U(N)_{-4}, \quad (1.11)$$

which gives⁴ (1.10). If we assume this, then our result (1.9) leads us to the Grassi-Hatsuda-Mariño relation (1.8), while if we assume (1.8), then our result (1.9) indicates the conjectural equivalence (1.10).

We also discuss that partition functions of \hat{A}_3 quiver theories have representations in terms of ideal Fermi gas systems associated with \hat{D} -type quivers⁵ and this leads an interesting relation between certain $U(N)$ and $USp(2N)$ SUSY gauge theories with single node. The $U(N)$ gauge theory under consideration is $\mathcal{N} = 4$ vector multiplet with one adjoint hyper multiplet and N_f fundamental hyper multiplets, while the $USp(2N)$ gauge theory is $\mathcal{N} = 4$ vector multiplet with one anti-symmetric hyper multiplet and N_f -fundamental hyper multiples. Regarding these theories, the work [35] has proposed the equivalence

$$Z_{U(N)+adj.}(N, N_f = 4) = Z_{USp(2N)+A}(N, N_f = 3). \quad (1.12)$$

This relation is expected from 3d mirror symmetry⁶ [42, 43]. It is known that the $U(N)$ and $USp(2N)$ theories are equivalent to \hat{A}_{N_f-1} and \hat{D}_{N_f} quiver theories without CS terms, where only one of the vector multiplets is coupled to one fundamental hyper multiplet. Since $\hat{A}_3 = \hat{D}_3$, (1.12) should hold via the 3d mirror symmetries. In appendix we explicitly prove this relation by using the technique in [27, 28].

This paper is organized as follows. In sec. 2, we consider the $\mathcal{N} = 5$ ABJM theory with the gauge group $O(2N+1)_{2k} \times USp(2N)_{-k}$. In sec. 3, we generalize our analysis in sec. 2 to more general quiver gauge theories. We also identify quantum mechanical operators in ideal Fermi gas systems naturally corresponding to orientifolds in type IIB brane constructions. As interesting examples, we deal with orientifold projections of the (p, q) model and orbifold ABJM theory. Section 4 is devoted to conclusion and discussions. In appendix, we explicitly prove the equivalence (1.12).

³ In order to fix the value of M , we should compare not only the moduli but also discrete torsion [9].

⁴ This statement has been partially checked by using superconformal index [41].

⁵ The papers [27, 28] have written partition functions of \hat{D} -type quiver theories in terms of ideal Fermi gas systems. it is unclear to us whether their derivation includes \hat{D}_3 case. However their derivation apparently seems to consider $\hat{D}_{n \geq 4}$ and it is unclear to us whether their derivation includes the \hat{D}_3 case or not. Hence we explicitly prove this for the \hat{D}_3 case. Even if [27, 28] did not prove it for the \hat{D}_3 case, our derivation is not essentially new.

⁶ We thank Kazumi Okuyama for useful discussions on this point.

Multiplet	One-loop determinant
$\mathcal{N} = 2$ $O(2N + 1)$ vector multiplet	$\prod_{i < j} \left[2 \sinh \frac{\mu_i - \mu_j}{2} \cdot 2 \sinh \frac{\mu_i + \mu_j}{2} \right]^2 \prod_{j=1}^N 4 \sinh^2 \frac{\mu_j}{2}$
$\mathcal{N} = 2$ $USp(2N)$ vector multiplet	$\prod_{i < j} \left[2 \sinh \frac{\nu_i - \nu_j}{2} \cdot 2 \sinh \frac{\nu_i + \nu_j}{2} \right]^2 \prod_{j=1}^N 4 \sinh^2 \nu_j$
$O(2N + 1) \times USp(2N)$ bi-fund. chiral mult.	$\left(\prod_{i,j} \left[2 \cosh \frac{\mu_i - \nu_j}{2} \cdot 2 \cosh \frac{\mu_i + \nu_j}{2} \right] \prod_j 2 \cosh \frac{\nu_j}{2} \right)^{-1}$

Table 1: One-loop determinant of each multiplet in the localization of the $O(2N + 1)_{2k} \times USp(2N)_{-k}$ ABJM theory on S^3 .

2 $O(2N + 1)_{2k} \times USp(2N)_{-k}$ ABJM theory

In this section we consider the $\mathcal{N} = 5$ ABJM theory with the gauge group $O(2N + 1)_{2k} \times USp(2N)_{-k}$. We will generalize our analysis in this section to more general theory in next section.

2.1 Orientifold ABJM theory as a Fermi gas

Thanks to the localization [2], the partition function of the $O(2N + 1)_{2k} \times USp(2N)_{-k}$ ABJM theory on S^3 can be written as⁷ (see tab. 1 for detail)

$$\begin{aligned}
Z_{\mathcal{N}=5\text{ABJM}}(N) &= \frac{1}{2^{2N} N!^2} \int \frac{d^N \mu}{(2\pi)^N} \frac{d^N \nu}{(2\pi)^N} e^{\frac{ik}{2\pi} \sum_{j=1}^N (\mu_j^2 - \nu_j^2)} \prod_{j=1}^N 4 \sinh^2 \frac{\mu_j}{2} \cdot 4 \sinh^2 \nu_j \\
&\times \frac{\prod_{i < j} \left[2 \sinh \frac{\mu_i - \mu_j}{2} \cdot 2 \sinh \frac{\mu_i + \mu_j}{2} \cdot 2 \sinh \frac{\nu_i - \nu_j}{2} \cdot 2 \sinh \frac{\nu_i + \nu_j}{2} \right]^2}{\prod_{i,j} \left[2 \cosh \frac{\mu_i - \nu_j}{2} \cdot 2 \cosh \frac{\mu_i + \nu_j}{2} \right]^2 \prod_j 4 \cosh^2 \frac{\nu_j}{2}}. \quad (2.1)
\end{aligned}$$

Now we write $Z_{\mathcal{N}=5\text{ABJM}}$ in terms of an ideal Fermi gas as in circular quiver $U(N)$ SUSY gauge theories [3, 14]. For this purpose we use the Cauchy determinant-like formula [26]

$$\begin{aligned}
&\frac{\prod_{i < j} \left[2 \sinh \frac{\mu_i - \mu_j}{2} \cdot 2 \sinh \frac{\mu_i + \mu_j}{2} \cdot 2 \sinh \frac{\nu_i - \nu_j}{2} \cdot 2 \sinh \frac{\nu_i + \nu_j}{2} \right]}{\prod_{i,j} \left[2 \cosh \frac{\mu_i - \nu_j}{2} \cdot 2 \cosh \frac{\mu_i + \nu_j}{2} \right]} \\
&= \sum_{\sigma \in S_N} (-1)^\sigma \prod_j \frac{1}{2 \cosh \left(\frac{\mu_j - \nu_{\sigma(j)}}{2} \right) \cdot 2 \cosh \left(\frac{\mu_j + \nu_{\sigma(j)}}{2} \right)}, \quad (2.2)
\end{aligned}$$

and rewrite the partition function as

$$Z_{\mathcal{N}=5\text{ABJM}}(N) \quad (2.3)$$

⁷ Note that the $O(2N + 1)_{2k} \times USp(2N)_{-k}$ $\mathcal{N} = 5$ ABJM theory has only one bi-fundamental hyper multiplet since one of two bi-fundamental hyper multiplets in the $\mathcal{N} = 6$ ABJM theory is projected out by the orientifold projection [38, 9]. In the localization formula, the $O \times USp$ bi-fundamental chiral multiplet (with R-charge 1/2) behaves like “half” of the hyper multiplet because of the group structure.

$$\begin{aligned}
&= \frac{1}{2^{2N} N!} \sum_{\sigma \in S_N} (-1)^\sigma \int \frac{d^N \mu}{(2\pi)^N} \frac{d^N \nu}{(2\pi)^N} e^{\frac{ik}{2\pi} \sum_{j=1}^N (\mu_j^2 - \nu_j^2)} \prod_{j=1}^N 4 \sinh^2 \frac{\mu_j}{2} \cdot 4 \sinh^2 \nu_j \\
&\quad \times \prod_j \frac{1}{2 \cosh \left(\frac{\mu_j - \nu_j}{2} \right) \cdot 2 \cosh \left(\frac{\mu_j + \nu_j}{2} \right) \cdot 2 \cosh \left(\frac{\mu_j - \nu_{\sigma(j)}}{2} \right) \cdot 2 \cosh \left(\frac{\mu_j + \nu_{\sigma(j)}}{2} \right) 4 \cosh^2 \frac{\nu_j}{2}} \\
&= \frac{1}{N!} \sum_{\sigma \in S_N} (-1)^\sigma \int d^N \mu \prod_j \rho_{\mathcal{N}=5\text{ABJM}}(\mu_j, \mu_{\sigma(j)}), \tag{2.4}
\end{aligned}$$

where

$$\begin{aligned}
\rho_{\mathcal{N}=5\text{ABJM}}(x, y) &= \frac{1}{2\pi k'} \sinh \frac{x}{2k'} \sinh \frac{y}{2k'} \\
&\quad \int \frac{d\nu}{2\pi k'} \frac{\sinh^2 \frac{\nu}{k'} \cdot e^{\frac{i}{4\pi k'}(x^2 - \nu^2)}}{2 \cosh \left(\frac{x - \nu}{2k'} \right) \cdot 2 \cosh \left(\frac{x + \nu}{2k'} \right) \cdot 2 \cosh \left(\frac{\nu - y}{2k'} \right) \cdot 2 \cosh \left(\frac{\nu + y}{2k'} \right) \cdot \cosh^2 \frac{\nu}{2k'}}, \tag{2.5}
\end{aligned}$$

with $k' = 2k$. This equation tells us that the partition function of the $\mathcal{N} = 5$ ABJM theory is described by the ideal Fermi gas system with the density matrix $\rho_{\mathcal{N}=5\text{ABJM}}(x, y)$. We regard $\rho_{\mathcal{N}=5\text{ABJM}}(x, y)$ as the matrix element of a quantum mechanical operator as in [3],

$$\rho_{\mathcal{N}=5\text{ABJM}}(x, y) = \frac{1}{\hbar} \langle x | \hat{\rho}_{\mathcal{N}=5\text{ABJM}}(\hat{Q}, \hat{P}) | y \rangle, \tag{2.6}$$

where

$$[\hat{Q}, \hat{P}] = i\hbar, \quad \hbar = 2\pi k' = 4\pi k. \tag{2.7}$$

The operator $\hat{\rho}_{\mathcal{N}=5\text{ABJM}}$ is defined as

$$\begin{aligned}
\hat{\rho}_{\mathcal{N}=5\text{ABJM}}(\hat{Q}, \hat{P}) &= \frac{1}{4} e^{\frac{i}{2\hbar} \hat{Q}^2} \frac{1 - \hat{R}}{2 \cosh \frac{\hat{P}}{2}} \frac{1}{2 \sinh \frac{\hat{Q}}{2k'}} e^{-\frac{i}{2\hbar} \hat{Q}^2} \frac{\sinh^2 \frac{\hat{Q}}{k'}}{2 \sinh \frac{\hat{Q}}{2k'} \cosh^2 \frac{\hat{Q}}{2k'}} \frac{1 - \hat{R}}{2 \cosh \frac{\hat{P}}{2}} \\
&= e^{\frac{i}{2\hbar} \hat{Q}^2} \frac{1}{2 \cosh \frac{\hat{P}}{2}} \frac{1 - \hat{R}}{2} e^{-\frac{i}{2\hbar} \hat{Q}^2} \frac{1}{2 \cosh \frac{\hat{P}}{2}} \frac{1 - \hat{R}}{2}, \tag{2.8}
\end{aligned}$$

where $\hat{R}|x\rangle = |-x\rangle$ and we have used

$$\frac{2 \sinh \frac{x}{2k'} \cdot 2 \sinh \frac{y}{2k'}}{2 \cosh \frac{x-y}{2k'} \cdot 2 \cosh \frac{x+y}{2k'}} = \langle x | \frac{1 - \hat{R}}{2 \cosh \frac{\hat{P}}{2}} | y \rangle. \tag{2.9}$$

By using the operator equations⁸ $e^{\frac{i}{2\hbar} \hat{Q}^2} f(\hat{P}) e^{-\frac{i}{2\hbar} \hat{Q}^2} = f(\hat{P} - \hat{Q})$ and $e^{\frac{i}{2\hbar} \hat{P}^2} g(\hat{Q}) e^{-\frac{i}{2\hbar} \hat{P}^2} = g(\hat{Q} + \hat{P})$, we simplify $\hat{\rho}_{\mathcal{N}=5\text{ABJM}}$ as

$$\hat{\rho}_{\mathcal{N}=5\text{ABJM}}(\hat{Q}, \hat{P}) = \frac{1}{2 \cosh \frac{\hat{Q} - \hat{P}}{2}} \frac{1}{2 \cosh \frac{\hat{P}}{2}} \frac{1 - \hat{R}}{2} = e^{\frac{i}{2\hbar} \hat{P}^2} \frac{1}{2 \cosh \frac{\hat{Q}}{2}} e^{-\frac{i}{2\hbar} \hat{P}^2} \frac{1}{2 \cosh \frac{\hat{P}}{2}} \frac{1 - \hat{R}}{2}. \tag{2.10}$$

⁸ Note also $\hat{R}f(\hat{Q}) = f(-\hat{Q})\hat{R}$, $\hat{R}f(\hat{P}) = f(-\hat{P})\hat{R}$ and $((1 - \hat{R})/2)^2 = (1 - \hat{R})/2$.

Performing the similarity transformation

$$\hat{\rho}_{\mathcal{N}=5\text{ABJM}}(\hat{Q}, \hat{P}) \rightarrow \left(\sqrt{2 \cosh \frac{\hat{Q}}{2}} e^{\frac{i}{2\hbar} \hat{P}^2} \right) \hat{\rho}_{\mathcal{N}=5\text{ABJM}}(\hat{Q}, \hat{P}) \left(\sqrt{2 \cosh \frac{\hat{Q}}{2}} e^{\frac{i}{2\hbar} \hat{P}^2} \right)^{-1}, \quad (2.11)$$

we obtain the following highly simplified expression

$$\hat{\rho}_{\mathcal{N}=5\text{ABJM}}(\hat{Q}, \hat{P}) = \frac{1}{\left(2 \cosh \frac{\hat{Q}}{2}\right)^{1/2}} \frac{1}{2 \cosh \frac{\hat{P}}{2}} \frac{1}{\left(2 \cosh \frac{\hat{Q}}{2}\right)^{1/2}} \frac{1 - \hat{R}}{2}. \quad (2.12)$$

Recalling that $\hat{\rho}$ for the $\mathcal{N} = 6$ ABJM theory with the gauge group $U(N)_{2k} \times U(N)_{-2k}$ is given by⁹

$$\hat{\rho}_{\mathcal{N}=6\text{ABJM}}(\hat{Q}, \hat{P}) = \frac{1}{\left(2 \cosh \frac{\hat{Q}}{2}\right)^{1/2}} \frac{1}{2 \cosh \frac{\hat{P}}{2}} \frac{1}{\left(2 \cosh \frac{\hat{Q}}{2}\right)^{1/2}}, \quad (2.13)$$

we finally obtain

$$\hat{\rho}_{\mathcal{N}=5\text{ABJM}}(\hat{Q}, \hat{P}) = \hat{\rho}_{\mathcal{N}=6\text{ABJM}}(\hat{Q}, \hat{P}) \frac{1 - \hat{R}}{2}. \quad (2.14)$$

This indicates that the density matrix operator $\hat{\rho}_{\mathcal{N}=5\text{ABJM}}$ of the $\mathcal{N} = 5$ ABJM theory is the projection of the one of the $\mathcal{N} = 6$ ABJM theory. Since the $\mathcal{N} = 5$ ABJM theory is the orientifold projection of the $\mathcal{N} = 6$ ABJM theory, presumably the operation of $(1 - \hat{R})/2$ to $\hat{\rho}_{\mathcal{N}=6\text{ABJM}}$ corresponds to the orientifold projection. It is interesting if one can understand this relation more precisely.

Remarks

1. The representation (2.14) of $\hat{\rho}_{\mathcal{N}=5\text{ABJM}}$ gives the matrix element

$$\rho_{\mathcal{N}=5\text{ABJM}}(x, y) = \frac{1}{2} \frac{1}{\sqrt{2 \cosh \frac{x}{2}}} \frac{\sinh \frac{x}{4k} \sinh \frac{y}{4k}}{\cosh \frac{x-y}{4k} \cosh \frac{x+y}{4k}} \frac{1}{\sqrt{2 \cosh \frac{y}{2}}}. \quad (2.15)$$

This gives the following representation of the partition function

$$Z_{\mathcal{N}=5\text{ABJM}}(N, k) = \frac{1}{2^N N!} \int \frac{d^N x}{(2\pi)^N} \prod_{j=1}^N \frac{4 \sinh^2 \frac{x_j}{2k}}{2 \cosh x_j} \frac{\prod_{i < j} \left[2 \sinh \frac{x_i - x_j}{2k} \cdot 2 \sinh \frac{x_i + x_j}{2k} \right]^2}{\prod_{i, j} \left[2 \cosh \frac{x_i - x_j}{2k} \cdot 2 \cosh \frac{x_i + x_j}{2k} \right]}, \quad (2.16)$$

where we have rescaled as $x \rightarrow 2x$. Let us compare this with the partition function of the $USp(2N)$ gauge theory with $\mathcal{N} = 4$ vector multiplet, one symmetric hyper multiplet and N_f -fundamental hyper multiples¹⁰ (called $USp + S$ theory in [26]):

$$Z_{USp+S}(N, N_f)$$

⁹ Note that the definition of \hbar in (2.7) is slightly different from the one usually used in Fermi gas systems associated with $U(N)$ CS theories.

¹⁰ When we go to the last line from the second line, we have used $\sinh^2 \mu_j = 4 \sinh^2 \frac{\mu_j}{2} \cosh^2 \frac{\mu_j}{2}$.

$$\begin{aligned}
&= \frac{1}{2^N N!} \int \frac{d^N \mu}{(2\pi)^N} \prod_{j=1}^N \frac{4 \sinh^2 \mu_j}{2 \cosh \mu_j (2 \cosh \frac{\mu_j}{2})^{2N_f}} \frac{\prod_{i < j} \left[2 \sinh \frac{\mu_i - \mu_j}{2} \cdot 2 \sinh \frac{\mu_i + \mu_j}{2} \right]^2}{\prod_{i,j} \cosh \frac{\mu_i - \mu_j}{2} \cdot \cosh \frac{\mu_i + \mu_j}{2}} \\
&= \frac{1}{2^N N!} \int \frac{d^N \mu}{(2\pi)^N} \prod_{j=1}^N \frac{4 \sinh^2 \frac{\mu_j}{2}}{2 \cosh \mu_j (2 \cosh \frac{\mu_j}{2})^{2N_f - 2}} \frac{\prod_{i < j} \left[2 \sinh \frac{\mu_i - \mu_j}{2} \cdot 2 \sinh \frac{\mu_i + \mu_j}{2} \right]^2}{\prod_{i,j} \cosh \frac{\mu_i - \mu_j}{2} \cdot \cosh \frac{\mu_i + \mu_j}{2}}.
\end{aligned} \tag{2.17}$$

Comparing this with (2.16), we easily see that the $\mathcal{N} = 5$ ABJM theory with $k = 1$ agrees with¹¹ the $USp + S$ theory with $N_f = 1$:

$$Z_{\mathcal{N}=5\text{ABJM}}(N, k = 1) = Z_{USp+S}(N, N_f = 1). \tag{2.18}$$

It is interesting if one can understand this relation by the brane constructions. Note that this result is essentially the same as the recent result in [35], which has shown the equivalence between the grand partition function of the $USp + S$ theory with $N_f = 1$ and $\Xi^{(-)}$ part of the $U(N)_2 \times U(N)_{-2}$ ABJM theory. Because of (2.14), our result is equivalent to this result: $\Xi_{O(2N+1)_2 \times USp(2N)_{-1}} = \Xi_{U(N)_2 \times U(N)_{-2}}^{(-)} = \Xi_{USp+S}(N_f = 1)$.

2. When we identify the quantum mechanical operator (2.8) associated with $\rho_{\mathcal{N}=5\text{ABJM}}(x, y)$, we could use the following identity once or twice instead of (2.9),

$$\frac{2 \cosh \frac{x}{2k} \cdot 2 \cosh \frac{y}{2k}}{2 \cosh \frac{x-y}{2k} \cdot 2 \cosh \frac{x+y}{2k}} = \langle x | \frac{1 + \hat{R}}{2 \cosh \frac{\hat{P}}{2}} | y \rangle. \tag{2.19}$$

Then the partition function $Z_{\mathcal{N}=5\text{ABJM}}$ is described by different representations of $\hat{\rho}$. If we use this identity and (2.9) just once by once, then we find

$$\begin{aligned}
\hat{\rho}'_{\mathcal{N}=5\text{ABJM}} &= \frac{1}{4} \frac{e^{\frac{i}{2\hbar} \hat{Q}^2}}{2 \cosh \frac{\hat{Q}}{2k}} \frac{1 + \hat{R}}{2 \cosh \frac{\hat{P}}{2}} \frac{e^{-\frac{i}{2\hbar} \hat{Q}^2}}{2 \cosh \frac{\hat{Q}}{2k}} \frac{1}{2 \sinh \frac{\hat{Q}}{2k}} \frac{\sinh^2 \frac{\hat{Q}}{k}}{\cosh^2 \frac{\hat{Q}}{2k}} \frac{1 - \hat{R}}{2 \cosh \frac{\hat{P}}{2}} \frac{1}{2 \sinh \frac{\hat{Q}}{2k}} \\
&= \frac{1}{4} \frac{1}{2 \cosh \frac{\hat{Q}}{2k}} \frac{1}{2 \cosh \frac{\hat{P}-\hat{Q}}{2}} \frac{1 + \hat{R}}{\cosh^2 \frac{\hat{Q}}{2k}} \frac{\sinh \frac{\hat{Q}}{k}}{2 \cosh \frac{\hat{P}}{2}} \frac{1 - \hat{R}}{2 \sinh \frac{\hat{Q}}{2k}},
\end{aligned} \tag{2.20}$$

while if we use (2.19) twice, then we get

$$\hat{\rho}''_{\mathcal{N}=5\text{ABJM}} = \frac{1}{4} \frac{e^{\frac{i}{2\hbar} \hat{Q}^2}}{2 \cosh \frac{\hat{Q}}{2k}} \frac{1 + \hat{R}}{2 \cosh \frac{\hat{P}}{2}} \frac{e^{-\frac{i}{2\hbar} \hat{Q}^2}}{2 \cosh \frac{\hat{Q}}{2k}} \frac{1}{2 \cosh \frac{\hat{Q}}{2k}} \frac{\sinh^2 \frac{\hat{Q}}{k}}{\cosh^2 \frac{\hat{Q}}{2k}} \frac{1 + \hat{R}}{2 \cosh \frac{\hat{P}}{2}} \frac{1}{2 \cosh \frac{\hat{Q}}{2k}}. \tag{2.21}$$

¹¹ We can also compare this with the $O(2N+1)$ gauge theory with $\mathcal{N} = 4$ vector multiplet, one symmetric hyper multiplet and N_f -fundamental hyper multiples ($O(2N+1) + S$ theory). Because of $Z_{O(2N+1)+S}(N, N_f) = Z_{USp+S}(N, N_f - 2)$, the relation (2.18) also shows $Z_{\mathcal{N}=5\text{ABJM}}(N, k = 1) = Z_{O(2N+1)+S}(N, N_f = 3)$.

To summarize, we have four different representations of $\hat{\rho}$ to describe the same partition function $Z_{\mathcal{N}=5\text{ABJM}}$:

$$\hat{\rho}_{\mathcal{N}=5\text{ABJM}} = \frac{1}{4} e^{\frac{i}{2\hbar} \hat{Q}^2} f_{\pm}(\hat{Q}) \frac{1 \pm \hat{R}}{2 \cosh \frac{\hat{P}}{2}} f_{\pm}(\hat{Q}) e^{-\frac{i}{2\hbar} \hat{Q}^2} \cdot \frac{\sinh^2 \frac{\hat{Q}}{k}}{\cosh^2 \frac{\hat{Q}}{2k}} f_{\pm}(\hat{Q}) \frac{1 \pm \hat{R}}{2 \cosh \frac{\hat{P}}{2}} f_{\pm}(\hat{Q}), \quad (2.22)$$

where we can freely choose “+” or “−” at every “ $f_{\pm}(1 \pm \hat{R})f_{\pm}$ ” and f_{\pm} is given by

$$f_{+}(Q) = \frac{1}{2 \cosh \frac{Q}{2k}}, \quad f_{-}(Q) = \frac{1}{2 \sinh \frac{Q}{2k}}. \quad (2.23)$$

In this paper we always choose “−” since taking “−” seems technically simpler.

2.2 Exact grand partition function for $k = 1$

Here we find the exact form of the grand partition function of the $O(2N+1)_{2k} \times USp(2N)_{-k}$ ABJM theory for $k = 1$. Grassi, Hatsuda and Mariño conjectured [39]

$$\Xi_{U(N)_2 \times U(N)_{-2}}^{(-)}(\mu) = \Xi_{U(N)_1 \times U(N)_{-1}}(\mu/2 + \pi i/2) \cdot \Xi_{U(N)_1 \times U(N)_{-1}}(\mu/2 - \pi i/2). \quad (2.24)$$

Combining this with our result (2.14), we immediately find (1.6)

$$\begin{aligned} \Xi_{O(2N+1)_2 \times USp(2N)_{-1}}(\mu) &= \Xi_{U(N)_2 \times U(N)_{-2}}^{(-)}(\mu) \\ &= \Xi_{U(N)_1 \times U(N)_{-1}}(\mu/2 + \pi i/2) \cdot \Xi_{U(N)_1 \times U(N)_{-1}}(\mu/2 - \pi i/2). \end{aligned}$$

The exact form (1.7) of the grand partition function $\Xi_{U(N)_1 \times U(N)_{-1}}$ was proposed as [40]

$$\begin{aligned} &\Xi_{U(N)_1 \times U(N)_{-1}}(\mu) \\ &= \exp \left[\frac{3\mu}{4} - \frac{3}{4} \log 2 + \mathcal{F}_1 + F_1^{\text{NS}} - \frac{1}{4\pi^2} \left(\mathcal{F}_0(\lambda) - \lambda \partial_{\lambda} \mathcal{F}_0(\lambda) + \frac{\lambda^2}{2} \partial_{\lambda}^2 \mathcal{F}_0(\lambda) \right) \right] \\ &\quad \times \left(\vartheta_2(\bar{\xi}/4, \bar{\tau}/4) + i \vartheta_1(\bar{\xi}/4, \bar{\tau}/4) \right), \end{aligned}$$

where $\vartheta_{1,2}$ is the Jacobi theta function¹² and¹³

$$\begin{aligned} \lambda &= \frac{\kappa^2}{8\pi} {}_3F_2 \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, \frac{3}{2}; -\frac{\kappa^4}{16} \right), \quad \kappa = e^{\mu} \\ \partial_{\lambda} \mathcal{F}_0(\lambda) &= \frac{\kappa^2}{4} G_{3,3}^{2,3} \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \middle| -\frac{\kappa^4}{16} \right) + \frac{\pi^2 i \kappa^2}{2} {}_3F_2 \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, \frac{3}{2}; -\frac{\kappa^4}{16} \right), \end{aligned}$$

¹² Their definitions are

$$\vartheta_1(v, \tau) = \sum_{n \in \mathbb{Z}} (-1)^{n-1/2} e^{\pi i(n+1/2)^2 \tau + 2\pi i(n+1/2)v}, \quad \vartheta_2(v, \tau) = \sum_{n \in \mathbb{Z}} e^{\pi i(n+1/2)^2 \tau + 2\pi i(n+1/2)v}.$$

¹³See [40] for details.

$$\begin{aligned}\partial_\lambda^2 \mathcal{F}_0(\lambda) &= -8\pi^3 i\bar{\tau}, \quad \bar{\xi} = \frac{i}{4\pi^3} (\lambda \partial_\lambda^2 \mathcal{F}_0(\lambda) - \partial_\lambda \mathcal{F}_0(\lambda)), \\ \mathcal{F}_1 &= -\log \eta(2\bar{\tau}) - \frac{1}{2} \log 2, \quad F_1^{\text{NS}} = \frac{1}{12} \log e^{-4\mu} - \frac{1}{24} \log 1 + 16e^{-4\mu}.\end{aligned}\tag{2.25}$$

In terms of (1.6), we can explicitly write the exact form of the grand partition function of the $O(2N+1)_2 \times USp(2N)_{-1}$ ABJM theory.

2.3 Grassi-Hatsuda-Mariño exact functional relation from geometry

Grassi, Hatsuda and Mariño conjectured the relation (1.8) on the grand canonical partition function of the ABJ theory [39]:

$$\Xi_{U(N+1)_4 \times U(N)_{-4}}(\mu) = \Xi_{U(N)_2 \times U(N)_{-2}}^{(-)}(\mu).$$

Physical interpretation of this relation has been unclear and therefore this relation has been considered as accidental. Now we give a physical interpretation on this relation. Let us compare this result with our result (1.5):

$$\Xi_{O(2N+1)_{2k} \times USp(2N)_{-k}}[\mu] = \Xi_{U(N)_{2k} \times U(N)_{-2k}}^{(-)}[\mu].$$

Plugging (1.9) into (1.8) leads us to

$$\Xi_{U(N)_4 \times U(N+1)_{-4} \text{ABJ}}[\mu] = \Xi_{O(2N+1)_2 \times USp(2N)_{-1}}[\mu].$$

This relation is equivalent to the conjecture in [9]. The $O(2N+1)_{2k} \times USp(2N)_{-k}$ ABJM theory is expected to be low energy effective theories of N M2-branes probing \mathbb{C}^4/\hat{D}_k with the binary dihedral group \hat{D}_k , whose action to the complex coordinate (z_1, z_2, z_3, z_4) of \mathbb{C}^4 is

$$(z_1, z_2, z_3, z_4) \sim e^{\frac{\pi i}{k}}(z_1, z_2, z_3, z_4) \sim (iz_2^*, -iz_1^*, iz_4^*, -iz_3^*).\tag{2.26}$$

Since \mathbb{C}^4/\hat{D}_k for $k=1$ is $\mathbb{C}^4/\mathbb{Z}_4$, the moduli of the $O(2N+1)_2 \times USp(2N)_{-1}$ ABJM theory become the same as the one of the $U(N+M)_4 \times U(N)_{-4}$ ABJ(M) theory. Therefore the work [9] conjectured that the $O(2N+1)_2 \times USp(2N)_{-1}$ ABJM theory has $\mathcal{N}=6$ SUSY and equivalent to the $U(N+1)_4 \times U(N)_{-4}$ ABJ theory (see [41] for the test by superconformal index):

$$O(2N+1)_2 \times USp(2N)_{-1} \leftrightarrow U(N+1)_4 \times U(N)_{-4}.$$

If we assume this, then our result (1.5) leads us to the Grassi-Hatsuda-Mariño relation (1.8), while if we assume the Grassi-Hatsuda-Mariño relation (1.8), then our result (1.5) indicates the conjecture (1.11).

3 Generalization

In this section we generalize our analysis in sec. 2 to a class of CS theory, which is circular quiver with the gauge group $[O(2N+1) \times USp(2N)]^r$ and bi-fundamental chiral multiplets one by one between nearest neighboring pairs of the gauge groups.

3.1 Fermi gas formalism

Let us consider the circular quiver CS theory with the gauge group $O(2N+1)_{2k_1} \times USp(2N)_{k'_1} \times \dots \times O(2N+1)_{2k_r} \times USp(2N)_{k'_r}$, where $O(2N+1)_{2k_a}$ and $USp(2N+1)_{2k'_a}$ are coupled to $N_f^{(a)}$ and $N_f'^{(a)}$ fundamental hyper multiplets, respectively. We parametrize the CS levels k_a, k'_a as $k_a = kn_a, k'_a = kn'_a$ with rational numbers n_a and n'_a . Applying the localization, the partition function becomes [2]

$$\begin{aligned} & Z_{O(2N+1) \times USp(2N)}(N) \\ &= \frac{1}{2^{2rN} N!^{2r}} \int \prod_{a=1}^r \frac{d^N \mu^{(a)}}{(2\pi)^N} \frac{d^N \nu^{(a)}}{(2\pi)^N} \prod_{j=1}^N 4 \sinh^2 \frac{\mu_j^{(a)}}{2} f^{(a)}(\mu_j^{(a)}) \cdot \frac{\sinh^2 \frac{\nu_j^{(a)}}{2} f'^{(a)}(\nu_j^{(a)})}{\cosh^2 \frac{\nu_j^{(a)}}{2}} \\ & \times \frac{\prod_{i < j} \left[2 \sinh \frac{\mu_i^{(a)} - \mu_j^{(a)}}{2} \cdot 2 \sinh \frac{\mu_i^{(a)} + \mu_j^{(a)}}{2} \cdot 2 \sinh \frac{\nu_i^{(a)} - \nu_j^{(a)}}{2} \cdot 2 \sinh \frac{\nu_i^{(a)} + \nu_j^{(a)}}{2} \right]^2}{\prod_{i,j} 2 \cosh \frac{\mu_i^{(a)} - \nu_j^{(a)}}{2} \cdot 2 \cosh \frac{\mu_i^{(a)} + \nu_j^{(a)}}{2} \cdot 2 \cosh \frac{\mu_i^{(a+1)} - \nu_j^{(a)}}{2} \cdot 2 \cosh \frac{\mu_i^{(a+1)} + \nu_j^{(a)}}{2}}, \end{aligned} \quad (3.1)$$

where $\mu_i^{(r+1)} = \mu_i^{(1)}$ and¹⁴

$$f^{(a)}(x) = \frac{e^{\frac{ik_a}{2\pi} x^2}}{(2 \cosh \frac{x}{2})^{2N_f^{(a)}}}, \quad f'^{(a)}(x) = \frac{e^{\frac{ik'_a}{2\pi} x^2}}{(2 \cosh \frac{x}{2})^{2N_f'^{(a)}}}. \quad (3.2)$$

By similar arguments to sec. 2.1, we rewrite the partition function as

$$Z_{O(2N+1) \times USp(2N)}(N) = \frac{1}{N!} \sum_{\sigma \in S_N} (-1)^\sigma \int d^N \mu \prod_j \rho(\mu_j, \mu_{\sigma(j)}).$$

Here the function $\rho(x, y)$ is defined by

$$\begin{aligned} & \rho_{O(2N+1) \times USp(2N)}(x, y) \\ &= \frac{\sinh \frac{x}{2k'} \sinh \frac{y}{2k'} f^{(1)}(x)}{2\pi k'} \int \left(\prod_{a=2}^r \frac{d\mu^{(a)}}{2\pi k'} f^{(a)}(\mu^{(a)}) \right) \left(\prod_{b=1}^r \frac{d\nu^{(b)} \sinh^2 \frac{\nu^{(b)}}{k'} f'^{(b)}(\nu^{(b)})}{2\pi k' \cosh^2 \frac{\nu^{(b)}}{2k'}} \right) \\ & \frac{1}{2 \cosh \left(\frac{x - \nu^{(1)}}{2k'} \right) \cdot 2 \cosh \left(\frac{x + \nu^{(1)}}{2k'} \right)} \frac{1}{\prod_{a=1}^{r-1} 2 \cosh \left(\frac{\nu^{(a)} - \mu^{(a+1)}}{2k'} \right) \cdot 2 \cosh \left(\frac{\nu^{(a)} + \mu^{(a+1)}}{2k'} \right)} \\ & \frac{1}{\prod_{a=2}^r 2 \cosh \left(\frac{\nu^{(a)} - \mu^{(a)}}{2k'} \right) \cdot 2 \cosh \left(\frac{\nu^{(a)} + \mu^{(a)}}{2k'} \right)} \frac{1}{2 \cosh \left(\frac{\nu^{(r)} - y}{2k'} \right) \cdot 2 \cosh \left(\frac{\nu^{(r)} + y}{2k'} \right)}. \end{aligned} \quad (3.3)$$

By appropriate similarity transformations, we obtain

$$\hat{\rho}_{O(2N+1) \times USp(2N)}(\hat{Q}, \hat{P}) = \prod_{a=1}^r f^{(a)}(\hat{Q}) \frac{1}{2 \cosh \frac{\hat{P}}{2}} \frac{1 - \hat{R}}{2} f'^{(a)}(\hat{Q}) \frac{1}{2 \cosh \frac{\hat{P}}{2}} \frac{1 - \hat{R}}{2}$$

¹⁴ We could also include masses and FI-terms. Then $f^{(a)}(x)$ and $f'^{(a)}(x)$ are modified but the result in this section does not essentially change up to this modification.

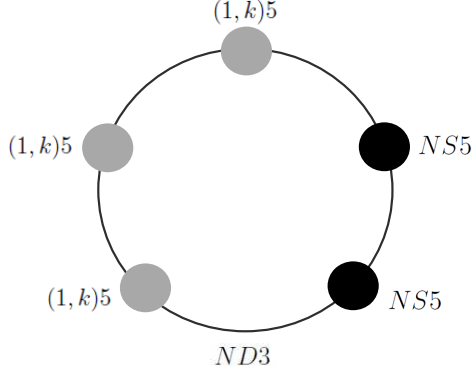


Figure 1: The type IIB brane construction of the (p, q) model for $(p, q) = (2, 3)$.

$$= \hat{\rho}_{U(N)}(\hat{Q}, \hat{P}) \frac{1 - \hat{R}}{2}, \quad (3.4)$$

where $\hat{\rho}_{U(N)}$ is the density matrix operator associated with the non-orientifold theory, which is obtained by the replacement $O(2N + 1), USp(2N) \rightarrow U(N)$ in the orientifold theories. This relation shows that $\hat{\rho}$ for the orientifold theory is the projection of the one without the orientifolds.

3.2 Identification of operators corresponding to orientifolds

Here we identify quantum mechanical operators, which naturally correspond to the orientifolds¹⁵ $\widetilde{O3}^{\pm}$ in type IIB brane construction. First, it is known that D5-brane, NS5-brane and $(1, k)$ -5 brane naturally correspond to¹⁶ (see e.g. [44, 45])

$$\hat{\mathcal{O}}_{D5} = \frac{1}{2 \cosh \frac{Q}{2}}, \quad \hat{\mathcal{O}}_{NS5} = \frac{1}{2 \cosh \frac{P}{2}}, \quad \hat{\mathcal{O}}_{(1,k)5} = e^{\frac{iQ^2}{2\hbar}} \frac{1}{2 \cosh \frac{P}{2}} e^{-\frac{iQ^2}{2\hbar}} = \frac{1}{2 \cosh \frac{P-Q}{2}}. \quad (3.5)$$

This is actually consistent with $\hat{\rho}$ of $\mathcal{N} = 3$ circular quiver CS theory with $U(N)$ gauge group and $SL(2, \mathbb{Z})$ symmetry in type IIB string. For example, let us consider the (p, q) -model, whose IIB brane construction consists of p NS5-branes and q $(1, k)$ -5 branes connected by N D3-branes on a circle (see fig. 1). This theory is $\mathcal{N} = 3$ circular quiver superconformal CS theory with the gauge group $U(N)_k \times U(N)_0^{q-1} \times U(N)_{-k}^{p-1}$ and $\hat{\rho}_{(p,q)}$ associated with this theory is

$$\hat{\rho}_{(p,q)} = \hat{\mathcal{O}}_{(1,k)5}^q \hat{\mathcal{O}}_{NS5}^p = \left(\frac{1}{2 \cosh \frac{P-Q}{2}} \right)^q \left(\frac{1}{2 \cosh \frac{P}{2}} \right)^p, \quad (3.6)$$

¹⁵ $\widetilde{O3}^-$ can be regarded as $O3^-$ plane with a half D3-brane while $\widetilde{O3}^+$ is perturbatively the same as $O3^+$ plane but different non-perturbatively.

¹⁶ We could also consider $(1, \tilde{k})$ -5 brane with $\tilde{k} = nk$, whose corresponding operator is $\hat{\mathcal{O}}_{(1,\tilde{k})5} = e^{\frac{i\tilde{n}Q^2}{2\hbar}} \frac{1}{2 \cosh \frac{P}{2}} e^{-\frac{i\tilde{n}Q^2}{2\hbar}} = \frac{1}{2 \cosh \frac{P-\tilde{n}Q}{2}}.$

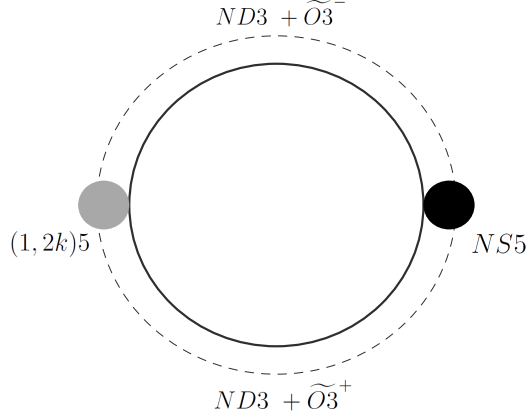


Figure 2: The type IIB brane construction of the $\mathcal{N} = 5$ ABJM theory with the gauge group $O(2N+1)_{2k} \times USp(2N)_{-k}$.

which is equivalent to $\hat{\rho}$ of the (p, q) -model by an appropriate canonical transformation.

Similarly let us consider the $O(2N+1)_{2k} \times USp(2N)_{-k}$ ABJM theory, whose brane construction is given by $(\widetilde{O3}^+ - D3) - (1, 2k) - (\widetilde{O3}^- - D3) - (NS5)$ on a circle (see fig. 2). As discussed in sec. 2.1, $\hat{\rho}$ for the $\mathcal{N} = 5$ ABJM theory is

$$\hat{\rho}_{\mathcal{N}=5\text{ABJM}} = e^{\frac{i}{2\hbar}\hat{Q}^2} \frac{1}{2 \cosh \frac{\hat{P}}{2}} \frac{1 - \hat{R}}{2} e^{-\frac{i}{2\hbar}\hat{Q}^2} \frac{1}{2 \cosh \frac{\hat{P}}{2}} \frac{1 - \hat{R}}{2} = \frac{1 - \hat{R}}{2} \hat{\mathcal{O}}_{(1,2k)} \frac{1 - \hat{R}}{2} \hat{\mathcal{O}}_{NS5}. \quad (3.7)$$

If we assume that this can be rewritten as

$$\hat{\rho}_{\mathcal{N}=5\text{ABJM}} = \hat{\mathcal{O}}_{\widetilde{O3}^+} \hat{\mathcal{O}}_{(1,2k)} \hat{\mathcal{O}}_{\widetilde{O3}^-} \hat{\mathcal{O}}_{NS5}, \quad (3.8)$$

where $\hat{\mathcal{O}}_{\widetilde{O3}^\pm}$ corresponds to $\widetilde{O3}^\pm$, then it is natural to identify¹⁷

$$\hat{\mathcal{O}}_{\widetilde{O3}^-} = \frac{1 - \hat{R}}{2}, \quad \hat{\mathcal{O}}_{\widetilde{O3}^+} = \frac{1 - \hat{R}}{2}. \quad (3.9)$$

This identification is consistent for more general quiver gauge theories described in sec. 3.1.

3.3 Orientifold projection of (p, q) -model

As an interesting example, we consider orientifold projection of the (p, q) -model analyzed well in [26, 33, 34, 22, 23, 24]. The (p, q) -model is the circular quiver theory with the

¹⁷ As mentioned in remark 2 of sec. 2.1, we have multiple representations of $\hat{\rho}$ to describe the same partition function. Then identifications of $\mathcal{O}_{\widetilde{O3}^\pm}$ are more generally

$$\mathcal{O}_{\widetilde{O3}^-} = 4 \sinh^2 \frac{\hat{Q}}{2k} f_\pm^2(\hat{Q}) \frac{1 \pm \hat{R}}{2}, \quad \mathcal{O}_{\widetilde{O3}^+} = \frac{\sinh^2 \frac{\hat{Q}}{k} f_\pm^2(\hat{Q})}{\cosh^2 \frac{\hat{Q}}{2k}} \frac{1 \pm \hat{R}}{2}.$$

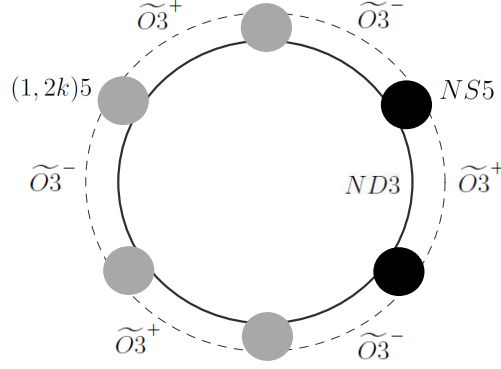


Figure 3: The type IIB brane construction of the orientifold projection of the (2, 4) model.

gauge group $U(N)_k \times U(N)_0^{q-1} \times U(N)_{-k} \times U(N)_0^{p-1}$, whose type IIB brane construction is $[(D3) - (NS5)]^p - [(D3) - (1, k)]^q$. Then let us consider a circular quiver theory with the brane construction (see fig. 3)

$$[(\widetilde{O3}^- - D3) - (NS5) - (\widetilde{O3}^+ - D3) - (NS5)]^p - [(\widetilde{O3}^- - D3) - (1, 2k) - (\widetilde{O3}^+ - D3) - (1, 2k)]^q.$$

Then corresponding $\hat{\rho}$ is

$$\begin{aligned} \hat{\rho} &= \left(\hat{\mathcal{O}}_{\widetilde{O3}^-}^{(-)} \hat{\mathcal{O}}_{NS5} \hat{\mathcal{O}}_{\widetilde{O3}^+}^{(-)} \hat{\mathcal{O}}_{NS5} \right)^p \left(\hat{\mathcal{O}}_{\widetilde{O3}^-}^{(-)} \hat{\mathcal{O}}_{(1, 2k)5} \hat{\mathcal{O}}_{\widetilde{O3}^+}^{(-)} \hat{\mathcal{O}}_{(1, 2k)5} \right)^q \\ &= \left(\frac{1}{2 \cosh \frac{\hat{P}}{2}} \right)^{2p} \left(\frac{1}{2 \cosh \frac{\hat{P}-\hat{Q}}{2}} \right)^{2q} \frac{1 - \hat{R}}{2} = \hat{\rho}_{(2p, 2q)} \frac{1 - \hat{R}}{2}. \end{aligned} \quad (3.10)$$

This can be understood as the projection of the $(2p, 2q)$ -model.

We can also consider the orientifold projection of the (p, q) -model with odd p and q . For example suppose the brane construction

$$[(\widetilde{O3}^- - D3) - (1, 2k) - (\widetilde{O3}^+ - D3) - (NS5)] - [(\widetilde{O3}^- - D3) - (NS5) - (\widetilde{O3}^+ - D3) - (NS5)]^m,$$

which gives

$$\begin{aligned} \hat{\rho} &= \hat{\mathcal{O}}_{\widetilde{O3}^-}^{(-)} \hat{\mathcal{O}}_{(1, 2k)5} \hat{\mathcal{O}}_{\widetilde{O3}^+}^{(-)} \hat{\mathcal{O}}_{NS5} \left(\hat{\mathcal{O}}_{\widetilde{O3}^-}^{(-)} \hat{\mathcal{O}}_{NS5} \hat{\mathcal{O}}_{\widetilde{O3}^+}^{(-)} \hat{\mathcal{O}}_{NS5} \right)^m \\ &= \frac{1}{2 \cosh \frac{\hat{P}-\hat{Q}}{2}} \left(\frac{1}{2 \cosh \frac{\hat{P}}{2}} \right)^{2m+1} \frac{1 - \hat{R}}{2} = \hat{\rho}_{(1, 2m+1)} \frac{1 - \hat{R}}{2}. \end{aligned} \quad (3.11)$$

This is the projection of the $(1, 2m + 1)$ -model.

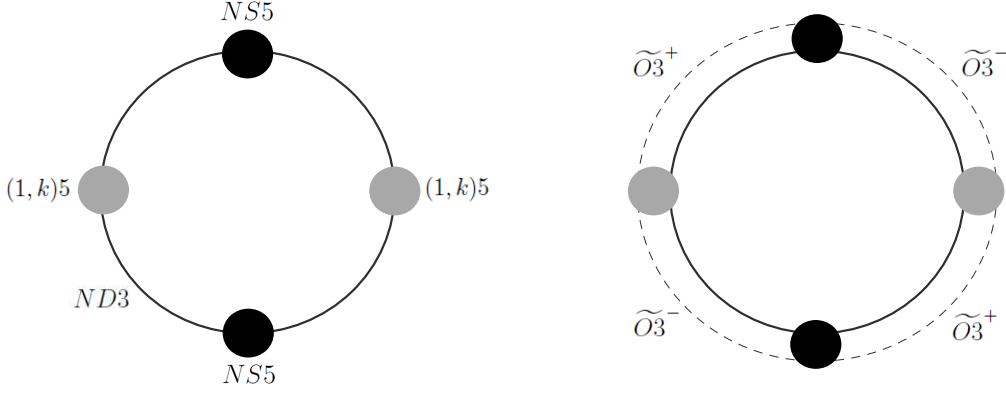


Figure 4: [Left] The type IIB brane construction of the orbifold ABJM theory for $r = 2$. [Right] Its orientifold projection.

3.4 Orientifold projection of orbifold ABJM theory

Next we consider the orientifold projection of the orbifold ABJM theory. Recalling that the brane construction of the orbifold ABJM theory is $[(D3) - (NS5) - (D3) - (1, k)]^r$, let us take the following brane construction (see fig. 4)

$$[(\widetilde{O3}^- - D3) - (NS5) - (\widetilde{O3}^+ - D3) - (1, 2k)]^r,$$

which gives the $[O(2N+1)_{2k} \times USp(2N)_{-k}]^r$ circular quiver superconformal CS theory. Then corresponding $\hat{\rho}$ is

$$\begin{aligned} \hat{\rho}_{[O(2N+1)_{2k} \times USp(2N)_{-k}]^r} &= \left[\mathcal{O}_{\widetilde{O3}^-}^{(-)} \mathcal{O}_{NS5} \mathcal{O}_{\widetilde{O3}^+}^{(-)} \mathcal{O}_{(1,2k)5} \right]^r \\ &= \left(\frac{1}{2 \cosh \frac{P}{2}} \frac{1}{2 \cosh \frac{P-Q}{2}} \right)^r \frac{1 - \hat{R}}{2} = (\hat{\rho}_{\mathcal{N}=5\text{ABJM}})^r, \end{aligned} \quad (3.12)$$

which is the projection of the orbifold ABJM theory. We can express the grand partition function of the orientifold projected orbifold ABJM theory in terms of the one of the $\mathcal{N} = 5$ ABJM theory by using the argument in [21]. Namely, when the density matrix operator $\hat{\rho}$ satisfies $\hat{\rho} = (\hat{\rho}')^r$, the grand partition function becomes

$$\text{Det}(1 + \rho e^\mu) = \prod_{j=-\frac{r-1}{2}}^{\frac{r-1}{2}} \text{Det}\left(1 + \rho' e^{\frac{\mu+2\pi i j}{r}}\right), \quad (3.13)$$

independent of detail form of $\hat{\rho}'$. Hence, the relation (3.12) immediately leads us to¹⁸

$$\Xi_{[O(2N+1)_{2k} \times USp(2N)_{-k}]^r}(\mu) = \prod_{j=-\frac{r-1}{2}}^{\frac{r-1}{2}} \Xi_{\mathcal{N}=5\text{ABJM}}\left(\frac{\mu + 2\pi i j}{r}\right). \quad (3.14)$$

¹⁸ Using the result of [21], we can also write “modified grand potential” of the orientifold projected orbifold ABJM theory in terms of the one of the $\mathcal{N} = 5$ ABJM theory.

Multiplet	One-loop determinant
$\mathcal{N} = 2$ $O(2N)$ vector multiplet	$\prod_{i < j} \left[2 \sinh \frac{\mu_i - \mu_j}{2} \cdot 2 \sinh \frac{\mu_i + \mu_j}{2} \right]^2$
$O(2N) \times USp(2N)$ bi-fund. chiral mult.	$\left(\prod_{i,j} \left[2 \cosh \frac{\mu_i - \nu_j}{2} \cdot 2 \cosh \frac{\mu_i + \nu_j}{2} \right] \right)^{-1}$

Table 2: One-loop determinant of each multiplet in the localization of the $O(2N)_{2k} \times USp(2N)_{-k}$ ABJM theory on S^3 .

Since we already know the exact form of the grand partition function for the $O(2N+1)_2 \times USp(2N)_{-1}$ by (1.6) and (1.7), we can also explicitly write the one of the orientifold projected orbifold ABJM theory with $k = 1$ in terms of (1.6).

3.5 Comments on $O(2N) \times USp(2N)$ type

In this section we give some comments on partition functions of $O(2N)_{2k} \times USp(2N)_{-k} \times \cdots \times O(2N)_{2k} \times USp(2N)_{-k}$ type theories, recently studied well in [37]. The S^3 partition function of this theory is technically equivalent to redefinition of $f^{(a)}(x)$ and $f'^{(a)}(x)$ in our analysis presented in sec. 3.1. For simplicity, let us consider the $\mathcal{N} = 5$ ABJM theory with the gauge group $O(2N)_{2k} \times USp(2N)_{-k}$. Applying the localization, the partition function of this theory becomes

$$\begin{aligned}
Z_{O(2N)_{2k} \times USp(2N)_{-k}} &= \frac{1}{2^{2N} N!^2} \int \frac{d^N \mu}{(2\pi)^N} \frac{d^N \nu}{(2\pi)^N} e^{\frac{ik'}{2\pi} \sum_{j=1}^N (\mu_j^2 - \nu_j^2)} \prod_{j=1}^N 4 \sinh^2 \nu_j \\
&\times \frac{\prod_{i < j} \left[2 \sinh \frac{\mu_i - \mu_j}{2} \cdot 2 \sinh \frac{\mu_i + \mu_j}{2} \cdot 2 \sinh \frac{\nu_i - \nu_j}{2} \cdot 2 \sinh \frac{\nu_i + \nu_j}{2} \right]^2}{\prod_{i,j} \left[2 \cosh \frac{\mu_i - \nu_j}{2} \cdot 2 \cosh \frac{\mu_i + \nu_j}{2} \right]^2}.
\end{aligned} \tag{3.15}$$

By similar arguments to sec. 3.1, we find

$$\begin{aligned}
&Z_{O(2N)_{2k} \times USp(2N)_{-k}} \\
&= \frac{1}{2^{2N} N!} \sum_{\sigma} (-1)^{\sigma} \int \frac{d^N \mu}{(2\pi)^N} \frac{d^N \nu}{(2\pi)^N} e^{\frac{ik}{2\pi} \sum_{j=1}^N (\mu_j^2 - \nu_j^2)} \prod_{j=1}^N 4 \sinh^2 \nu_j \\
&\times \prod_j \frac{1}{2 \cosh \left(\frac{\mu_j - \nu_j}{2} \right) \cdot 2 \cosh \left(\frac{\mu_j + \nu_j}{2} \right) \cdot 2 \cosh \left(\frac{\mu_j - \nu_{\sigma(j)}}{2} \right) \cdot 2 \cosh \left(\frac{\mu_j + \nu_{\sigma(j)}}{2} \right)} \\
&= \sum_{\sigma} (-1)^{\sigma} \int d^N \mu \prod_j \rho_{O(2N)_{2k} \times USp(2N)_{-k}}(\mu_j, \mu_{\sigma(j)}),
\end{aligned} \tag{3.16}$$

where

$$\rho_{O(2N)_{2k} \times USp(2N)_{-k}}(x, y) = \frac{1}{2\pi k} \int \frac{d\nu}{2\pi k} \frac{e^{\frac{i}{4\pi k}(x^2 - \nu^2)} \sinh^2 \frac{\nu}{k}}{2 \cosh\left(\frac{x-\nu}{2k}\right) \cdot 2 \cosh\left(\frac{x+\nu}{2k}\right) \cdot 2 \cosh\left(\frac{\nu-y}{2k}\right) \cdot 2 \cosh\left(\frac{\nu+y}{2k}\right)}. \quad (3.17)$$

The quantum mechanical operator $\hat{\rho}_{O(2N)_{2k} \times USp(2N)_{-k}}$ associated with this is

$$\hat{\rho}_{O(2N)_{2k} \times USp(2N)_{-k}} = e^{\frac{i}{2\hbar}\hat{Q}^2} f_{\pm}(\hat{Q}) \frac{1 \pm \hat{R}}{2 \cosh \frac{\hat{P}}{2}} f_{\pm}(\hat{Q}) e^{-\frac{i}{2\hbar}\hat{Q}^2} \cdot \sinh^2 \frac{\hat{Q}}{k} f_{\pm}(\hat{Q}) \frac{1 \pm \hat{R}}{2 \cosh \frac{\hat{P}}{2}} f_{\pm}(\hat{Q}), \quad (3.18)$$

which is of course the same as the result of [37].

Next we consider operators corresponding to orientifolds $O3^{\pm}$. Let us recall that the brane construction of the $O(2N)_{2k'} \times USp(2N)_{-k'}$ ABJM is given by $(O3^- - D3) - (NS5) - (O3^+ - D3) - (1, 2k')$. The $\hat{\rho}$ for the $O(2N)_{2k} \times USp(2N)_{-k}$ ABJM theory (3.18) can be rewritten as

$$\hat{\rho}_{O(2N)_{2k} \times USp(2N)_{-k}} = f_{\pm}(\hat{Q}) \frac{1 \pm \hat{R}}{2} \mathcal{O}_{(1,2k)} f_{\pm}(\hat{Q}) \cdot 4 \sinh^2 \frac{\hat{Q}}{k} f_{\pm}(\hat{Q}) \frac{1 \pm \hat{R}}{2} \mathcal{O}_{NS5} f_{\pm}(\hat{Q}). \quad (3.19)$$

Assuming $\hat{\rho}_{O(2N)_{2k} \times USp(2N)_{-k}} = \mathcal{O}_{O3^+} \mathcal{O}_{(1,2k)} \mathcal{O}_{O3^-} \mathcal{O}_{NS5}$, we arrive at the following identification

$$\mathcal{O}_{O3^+}^{(\pm)} = 4 \sinh^2 \frac{\hat{Q}}{k} f_{\pm}^2(\hat{Q}) \frac{1 \pm \hat{R}}{2}, \quad \mathcal{O}_{O3^-}^{(\pm)} = f_{\pm}^2(\hat{Q}) \frac{1 \pm \hat{R}}{2}. \quad (3.20)$$

4 Conclusion and Discussions

In this paper we have studied the partition functions of the low-energy effective theories of M2-branes, whose type IIB brane constructions include the orientifolds. We have mainly focused on the circular quiver superconformal CS theory on S^3 with the gauge group $O(2N+1) \times USp(2N) \times \cdots \times O(2N+1) \times USp(2N)$, which is the natural generalization of the $O(2N+1)_{2k} \times USp(2N)_{-k}$ $\mathcal{N} = 5$ ABJM theory. We have found that the partition function of this type of theory have the simple relation (1.5) to the one of the M2-brane theories without the orientifolds with the gauge group $U(N) \times \cdots \times U(N)$. By using this relation and the recent results in [39, 40], we have found the exact form (1.6) of the grand partition function of the $O(2N+1)_2 \times USp(2N)_{-1}$ ABJM theory, where its SUSY is expected to be enhanced to $\mathcal{N} = 6$ [9]. As another application, we discussed that our result gives the natural physical interpretation of the relation (1.8) conjectured by Grassi-Hatsuda-Mariño. We also argued in appendix that the partition function of \hat{A}_3 quiver theory has the representation (A.10) in terms of an ideal Fermi gas system of \hat{D} -type quiver theory and this leads the relation (1.12) between the $U(N)$ and $USp(2N)$ SUSY gauge theories.

Our result (1.2), (3.4) shows that the density matrix operator for the orientifold theory is the projection of the non-orientifold theory by the operator $(1 - \hat{R})/2$. It is nice if we can understand this relation more precisely. Our result also implies that one can systematically

study the partition function of the orientifold theory by using techniques developed in the studies of the non-orientifold theory. For instance the technique introduced in [35] allows us to compute WKB expansion of $\text{Tr}(\hat{\rho}^\ell \hat{R})$ systematically¹⁹ in terms of information on Wigner transformation of $\hat{\rho}^\ell$. It is interesting to determine non-perturbative effects in the orientifold theories by such techniques.

Recalling that the $U(N)_k \times U(N)_{-k}$ $\mathcal{N} = 6$ ABJM theory is described by topological string on local $\mathbb{P}^1 \times \mathbb{P}^1$, this relation would imply that the $O(2N+1)_{2k} \times USp(2N)_{-k}$ ABJM theory is described by certain projection in the topological string. There should be a physical meaning of $(1 - \hat{R})/2$ in the context of the topological string.

Although we have found the physical interpretation of one of relations conjectured by Grassi-Hatsuda-Mariño [39], they also conjectured other relations among the grand partition functions of the ABJ(M) theory with specific values of the parameters:

$$\begin{aligned} \Xi_{U(N)_4 \times U(N)_{-4}}(\mu) &= \Xi_{U(N+1)_2 \times U(N)_{-2}}^{(+)}(\mu), & \Xi_{U(N+2)_4 \times U(N)_{-4}}(\mu) &= \Xi_{U(N+1)_2 \times U(N)_{-2}}^{(-)}(\mu), \\ \Xi_{U(N+2)_8 \times U(N)_{-8}}(\mu) &= \Xi_{U(N+2)_4 \times U(N)_{-4}}^{(-)}(\mu). \end{aligned} \quad (4.1)$$

Although these relations might be accidental coincidences, it would be illuminating if we can find some physical interpretations.

One of immediate extensions of our analysis is to consider the gauge group $O(2N_1+1) \times USp(2N_2) \times \cdots \times O(2N_1+1) \times USp(2N_2)$. Probably this can be done by combining the technique in [20, 5] with the Cauchy determinant-like formula (2.2). If this is the case, $\hat{\rho}$ for the $O(2N_1+1)_{2k} \times USp(2N_2)_{-2k}$ $\mathcal{N} = 5$ ABJ(M) theory would be projection²⁰ of the one of the $U(N_1)_{2k} \times U(N_2)_{-2k}$ $\mathcal{N} = 6$ ABJ(M) theory by $(1 - \hat{R})/2$. Another interesting direction is to study other supersymmetric observables such as supersymmetric Wilson loops. Then the techniques established in [7] would be efficient.

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A An exact relation between $USp(2N)$ and $U(N)$ gauge theories

In this appendix we show the exact relation (1.12) between the SUSY gauge theories with $U(N)$ and $USp(2N)$ gauge groups. The $U(N)$ gauge theory, which we consider here, is $\mathcal{N} = 4$ vector multiplet with one adjoint hyper multiplet and N_f fundamental hyper multiplets,

¹⁹ We thank Kazumi Okuyama for discussions on this point.

²⁰ After this paper appeared in arXiv, this statement is proven in [46].

whose partition function is described by so-called N_f -matrix model [33]:

$$Z_{U+adj.}(N, N_f) = \frac{1}{N!} \int \frac{d^N \mu}{(2\pi)^N} \prod_{j=1}^N \frac{1}{(2 \cosh \frac{\mu_j}{2})^{N_f}} \prod_{i < j} \tanh^2 \frac{\mu_i - \mu_j}{2}. \quad (\text{A.1})$$

This matrix model has been studied well in [26, 33, 22, 23, 24]. The $USp(2N)$ gauge theory is $\mathcal{N} = 4$ vector multiplet with one anti-symmetric hyper multiplet and N_f -fundamental hyper multiples and its partition function is

$$Z_{USp+A}(N, N_f) = \frac{1}{2^{2N} N!} \int \frac{d^N \mu}{(2\pi)^N} \prod_{j=1}^N \frac{4 \sinh^2 \mu_j}{(4 \cosh^2 \frac{\mu_j}{2})^{N_f}} \prod_{i < j} \left[\frac{\sinh \frac{\mu_i - \mu_j}{2} \cdot \sinh \frac{\mu_i + \mu_j}{2}}{\cosh \frac{\mu_i - \mu_j}{2} \cdot \cosh \frac{\mu_i + \mu_j}{2}} \right]^2, \quad (\text{A.2})$$

which has been analyzed in [26, 27, 35]. Regarding these theories, the work [35] has proposed the following equivalence²¹

$$Z_{U(N)+adj.}(N, N_f = 4) = Z_{USp(2N)+A}(N, N_f = 3).$$

This relation is expected from 3d mirror symmetry [42, 43]. It is known that the $U(N)$ and $USp(2N)$ theories are equivalent to \hat{A}_{N_f-1} and \hat{D}_{N_f} quiver theories without CS levels, where only one of the vector multiples is coupled to one fundamental hyper multiplet, respectively. Since $\hat{A}_3 = \hat{D}_3$, the equation (1.12) should hold. In this section we explicitly prove this relation by using the technique in [27, 28].

A.1 $\hat{A}_3 = \hat{D}_3$

Here we show that partition function of \hat{A}_3 quiver theory has a representation in terms of an ideal Fermi gas system of \hat{D} -type quiver theory. Although this may be already proven in [27, 28], it is unclear to us whether their derivation includes our analysis in this section or not and therefore we explicitly prove it.

First we precisely explain what we would like to prove. Suppose the SUSY CS theory with \hat{A}_n quiver, namely the circular quiver with the gauge group $U(N)_{k_1} \times \cdots U(N)_{k_{n+1}}$, which is coupled to $N_f^{(a)}$ fundamental hyper multiplets. The partition function of the \hat{A}_n quiver theory can be denoted by [2]

$$Z_{\hat{A}_n} = \frac{1}{N!^{n+1}} \int \prod_{a=1}^{n+1} \frac{d^N \mu^{(a)}}{(2\pi)^N} \prod_{j=1}^N f^{(a)}(\mu_j^{(a)}) \frac{\prod_{i < j} \left[2 \sinh \frac{\mu_i^{(a)} - \mu_j^{(a)}}{2} \cdot 2 \sinh \frac{\mu_i^{(a)} + \mu_j^{(a)}}{2} \right]^2}{\prod_{i,j} 2 \cosh \frac{\mu_i^{(a)} - \mu_j^{(a+1)}}{2}},$$

²¹ One can also compare this with the $O(2N+1)$ gauge theory with $\mathcal{N} = 4$ vector multiplet, one anti-symmetric hyper multiplet and N_f -fundamental hyper multiples ($O(2N+1) + A$ theory). Then because of $Z_{O(2N+1)+A}(N, N_f) = Z_{USp+A}(N, N_f - 2)$, the relation (1.12) also indicates $Z_{U(N)+adj.}(N, N_f = 4) = Z_{O(2N+1)+A}(N, N_f = 5)$.

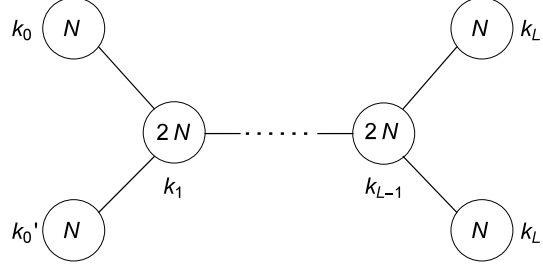


Figure 5: The \hat{D}_{L+2} quiver diagram.

where $\mu_j^{(n+2)} = \mu_j^{(1)}$ and

$$f^{(a)}(x) = \frac{e^{\frac{ik_a}{2\pi}x^2}}{(2 \cosh \frac{x}{2})^{2N_f^{(a)}}}. \quad (\text{A.3})$$

It is known that one can rewrite the partition function of the \hat{A}_n theory as [3]

$$Z_{\hat{A}_n}(N) = \sum_{\sigma \in S_N} (-1)^\sigma \int d^N x \prod_{j=1}^N \rho_{\hat{A}_n}(x_j, x_{\sigma(j)}), \quad (\text{A.4})$$

where $\rho_{\hat{A}_n}(x, y)$ is the density matrix of the ideal Fermi gas system associated with the quantum mechanical operator

$$\hat{\rho}_{\hat{A}_n}(\hat{Q}, \hat{P}) = \prod_{a=1}^{n+1} f^{(a)}(\hat{Q}) \frac{1}{2 \cosh \frac{\hat{P}}{2}}. \quad (\text{A.5})$$

Next let us consider the \hat{D}_{L+2} quiver CS theory with the gauge group $U(N)_{k_0} \times U(N)_{k'_0} \times U(2N)_{k_1} \times \cdots \times U(2N)_{k_{L-1}} \times U(N)_{k_L} \times U(N)_{k'_L}$ (see fig. 5). The partition function of this theory is given by [2]

$$\begin{aligned} Z_{\hat{D}_{L+2}} = & \frac{1}{N!^4 (2N!)^{L-1}} \int \frac{d^N \mu^{(0)}}{(2\pi)^N} \frac{d^N \mu'^{(0)}}{(2\pi)^N} \frac{d^N \mu^{(L)}}{(2\pi)^N} \frac{d^N \mu'^{(L)}}{(2\pi)^N} \prod_{a=1}^{L-1} \frac{d^{2N} \mu^{(a)}}{(2\pi)^{2N}} \prod_{a=1}^{L-1} \prod_{J=1}^{2N} F^{(a)}(\mu_J^{(a)}) \\ & \prod_{j=1}^N F^{(0)}(\mu_j^{(0)}) F'^{(0)}(\mu_j'^{(0)}) F^{(L)}(\mu_j^{(L)}) F'^{(L)}(\mu_j'^{(L)}) \frac{\prod_{a=1}^{L-1} \prod_{I \neq J} 2 \sinh \frac{\mu_I^{(a)} - \mu_J^{(a)}}{2}}{\prod_{a=1}^{L-2} \prod_{I, J} 2 \cosh \frac{\mu_I^{(a)} - \mu_J^{(a+1)}}{2}} \\ & \frac{\prod_{i \neq j} 2 \sinh \frac{\mu_i^{(0)} - \mu_j^{(0)}}{2} \cdot 2 \sinh \frac{\mu_i'^{(0)} - \mu_j'^{(0)}}{2} \cdot 2 \sinh \frac{\mu_i^{(L)} - \mu_j^{(L)}}{2} \cdot 2 \sinh \frac{\mu_i'^{(L)} - \mu_j'^{(L)}}{2}}{\prod_{i=1}^N \prod_{J=1}^{2N} 2 \cosh \frac{\mu_i^{(0)} - \mu_J^{(1)}}{2} \cdot 2 \cosh \frac{\mu_i'^{(0)} - \mu_J^{(1)}}{2} \cdot 2 \cosh \frac{\mu_i^{(L)} - \mu_J^{(L-1)}}{2} \cdot 2 \cosh \frac{\mu_i'^{(L)} - \mu_J^{(L-1)}}{2}}, \end{aligned} \quad (\text{A.6})$$

where

$$F^{(a)}(x) = \frac{e^{\frac{ika}{2\pi}x^2}}{(2 \cosh \frac{x}{2})^{2N_f^{(a)}}}, \quad F'^{(a)}(x) = \frac{e^{\frac{ik'_a}{2\pi}x^2}}{(2 \cosh \frac{x}{2})^{2N_f'^{(a)}}}. \quad (\text{A.7})$$

It is also known that the partition function of the \hat{D}_{L+2} quiver theory is described by an ideal Fermi gas system [27, 28]:

$$Z_{\hat{D}_{L+2}} = \frac{1}{N!} \sum_{\sigma \in S_N} (-1)^\sigma \int \frac{d^N x}{(2\pi)^N} \prod_{j=1}^N \rho_{\hat{D}_{L+2}}^{(\pm)}(x_j, x_{\sigma(j)}), \quad (\text{A.8})$$

where

$$\begin{aligned} & \hat{\rho}_{\hat{D}_{L+2}} \\ &= \left(F^{(0)}(\hat{Q}) \tanh \frac{\hat{P}}{2} F'^{(0)}(\hat{Q}) + F'^{(0)}(\hat{Q}) \tanh \frac{\hat{P}}{2} F^{(0)}(\hat{Q}) \right) \frac{1}{2 \cosh \frac{\hat{P}}{2}} \left(\prod_{a=1}^{L-1} F^{(a)}(\hat{Q}) \frac{1}{2 \cosh \frac{\hat{P}}{2}} \right) \\ & \left(F^{(L)}(\hat{Q}) \tanh \frac{\hat{P}}{2} F'^{(L)}(\hat{Q}) + F'^{(L)}(\hat{Q}) \tanh \frac{\hat{P}}{2} F^{(L)}(\hat{Q}) \right) \frac{1}{2 \cosh \frac{\hat{P}}{2}} \left(\prod_{a=1}^{L-1} F^{(L-a)}(\hat{Q}) \frac{1}{2 \cosh \frac{\hat{P}}{2}} \right). \end{aligned} \quad (\text{A.9})$$

In this section we prove

$$Z_{\hat{A}_3} = \frac{1}{N!} \sum_{\sigma \in S_N} (-1)^\sigma \int \frac{d^N x}{(2\pi)^N} \prod_{j=1}^N \left[\lim_{L \rightarrow 1} \rho_{\hat{D}_{L+2}}^{(\pm)}(x_j, x_{\sigma(j)}) \right], \quad (\text{A.10})$$

where

$$\begin{aligned} \lim_{L \rightarrow 1} \hat{\rho}_{\hat{D}_{L+2}} &= \left(F^{(0)}(\hat{Q}) \tanh \frac{\hat{P}}{2} F'^{(0)}(\hat{Q}) + F'^{(0)}(\hat{Q}) \tanh \frac{\hat{P}}{2} F^{(0)}(\hat{Q}) \right) \frac{1}{2 \cosh \frac{\hat{P}}{2}} \\ & \left(F^{(1)}(\hat{Q}) \tanh \frac{\hat{P}}{2} F'^{(1)}(\hat{Q}) + F'^{(1)}(\hat{Q}) \tanh \frac{\hat{P}}{2} F^{(1)}(\hat{Q}) \right) \frac{1}{2 \cosh \frac{\hat{P}}{2}}. \end{aligned} \quad (\text{A.11})$$

As mentioned above, this may be already proven in [27, 28]. However their derivation apparently seems to take $L \geq 2$, where at least one $U(2N)$ node is present, and it is unclear to us whether their derivation includes \hat{D}_3 ($L = 1$) case or not. Therefore we explicitly prove this relation.

Now let us consider the A_3 quiver theory:

$$Z_{\hat{A}_3} = \frac{1}{N!^4} \int \prod_{a=1}^4 \frac{d^N \mu^{(a)}}{(2\pi)^N} \prod_{j=1}^N f^{(a)}(\mu_j^{(a)}) \frac{\prod_{i < j} \left[2 \sinh \frac{\mu_i^{(a)} - \mu_j^{(a)}}{2} \cdot 2 \sinh \frac{\mu_i^{(a)} + \mu_j^{(a)}}{2} \right]^2}{\prod_{i,j} 2 \cosh \frac{\mu_i^{(a)} - \mu_j^{(a+1)}}{2}}.$$

Let us redefine the variables as in \hat{D}_3 -quiver language:

$$\mu_j^{(1)} = x_j, \quad \mu_j^{(2)} = y_{N+j}, \quad \mu_j^{(3)} = x_{N+j}, \quad \mu_j^{(4)} = y_j,$$

$$F^{(0)}(x) = f^{(1)}(x), \quad F'^{(0)}(x) = f^{(3)}(x), \quad F^{(1)}(x) = f^{(4)}(x), \quad F'^{(1)}(x) = f^{(2)}(x). \quad (\text{A.12})$$

Then the partition function becomes

$$Z_{\hat{A}_3} = \frac{1}{N!^4} \int \frac{d^{2N}x}{(2\pi)^{2N}} \frac{d^{2N}y}{(2\pi)^{2N}} \prod_{j=1}^N F^{(0)}(x_j) F'^{(0)}(x_{N+j}) F^{(1)}(y_j) F'^{(1)}(y_{N+j}) \frac{\left[\prod_{i < j} 2 \sinh \frac{x_i - x_j}{2} \cdot 2 \sinh \frac{x_{N+i} - x_{N+j}}{2} \cdot 2 \sinh \frac{y_i - y_j}{2} \cdot 2 \sinh \frac{y_{N+i} - y_{N+j}}{2} \right]^2}{\prod_{I,J} 2 \cosh \frac{x_I - y_J}{2}}, \quad (\text{A.13})$$

where $I, J = 1, \dots, 2N$. By inserting

$$1 = \frac{\prod_{i,j} 2 \sinh \frac{x_i - x_{N+j}}{2} \cdot 2 \sinh \frac{y_i - y_{N+j}}{2}}{\prod_{i,j} 2 \sinh \frac{x_i - x_{N+j}}{2} \cdot 2 \sinh \frac{y_i - y_{N+j}}{2}}, \quad (\text{A.14})$$

to the integrand and using the Cauchy determinant formula, we find

$$Z_{\hat{A}_3} = \frac{1}{N!^2} \sum_{\sigma \in S_{2N}} (-1)^\sigma \int \frac{d^{2N}x}{(2\pi)^{2N}} \frac{d^{2N}y}{(2\pi)^{2N}} \prod_{j=1}^N F^{(0)}(x_j) F'^{(0)}(x_{N+j}) F^{(1)}(y_j) F'^{(1)}(y_{N+j}) \times \frac{1}{\prod_{j=1}^N 2 \sinh \frac{x_j - x_{N+j}}{2} \cdot 2 \sinh \frac{y_j - y_{N+j}}{2}} \frac{1}{\prod_{J=1}^{2N} 2 \cosh \frac{x_J - y_{\sigma(J)}}{2}}. \quad (\text{A.15})$$

Below in this subsection we just repeat the argument of [27]. According to [27], we introduce

$$R(j) = N + j, \quad R(N + j) = j. \quad (\text{A.16})$$

Now we would like to rewrite the integral in terms of a kernel acting on set of N eigenvalues $\mathcal{K}(\sigma)$ among x_j 's, which is dependent on the permutation σ . More precisely, we take $\mathcal{K}(\sigma)$ such that $R\tau^{-1}R\tau(j) \in \mathcal{K}(\sigma)$ for given $j \in \mathcal{K}(\sigma)$. Then we rewrite the partition function as

$$Z_{\hat{A}_3} = \frac{1}{N!^2} \sum_{\sigma \in S_{2N}} (-1)^\sigma \int \frac{d^{2N}x}{(2\pi)^{2N}} \frac{d^{2N}y}{(2\pi)^{2N}} \prod_{j=1}^N F^{(0)}(x_j) F'^{(0)}(x_{N+j}) F^{(1)}(y_j) F'^{(1)}(y_{N+j}) \prod_{j \in \mathcal{K}(\sigma)} \frac{1}{2 \cosh \frac{x_j - y_{\sigma(j)}}{2}} \frac{(-1)^{s(\sigma(j))}}{2 \sinh \frac{y_{\sigma(j)} - y_{R\sigma(j)}}{2}} \frac{1}{2 \cosh \frac{y_{R\sigma(j)} - x_{\sigma^{-1}R\sigma(j)}}{2}} \frac{(-1)^{s(\sigma^{-1}R\sigma(j))}}{2 \sinh \frac{x_{\sigma^{-1}R\sigma(j)} - x_{R\sigma^{-1}R\sigma(j)}}{2}}, \quad (\text{A.17})$$

where

$$s(j) = \begin{cases} 0 & \text{for } j = 1, \dots, N \\ 1 & \text{for } j = N + 1, \dots, 2N \end{cases}. \quad (\text{A.18})$$

Note that we can also write this as

$$Z_{\hat{A}_3} = \frac{1}{N!^2} \sum_{\sigma \in S_{2N}} (-1)^{R\sigma} \int \frac{d^{2N}x}{(2\pi)^{2N}} \frac{d^{2N}y}{(2\pi)^{2N}} \prod_{j=1}^N F^{(0)}(x_j) F'^{(0)}(x_{N+j}) F^{(1)}(y_j) F'^{(1)}(y_{N+j})$$

$$\prod_{j \in \mathcal{K}(\sigma)} \frac{1}{2 \cosh \frac{x_j - y_{R\sigma(j)}}{2}} \frac{(-1)^{s(R\sigma(j))}}{2 \sinh \frac{y_{R\sigma(j)} - y_{\sigma(j)}}{2}} \frac{1}{2 \cosh \frac{y_{\sigma(j)} - x_{\sigma^{-1}R\sigma(j)}}{2}} \frac{(-1)^{s(\sigma^{-1}R\sigma(j))}}{2 \sinh \frac{x_{\sigma^{-1}R\sigma(j)} - x_{R\sigma^{-1}R\sigma(j)}}{2}}. \quad (\text{A.19})$$

Averaging over these, we obtain

$$\begin{aligned} Z_{\hat{A}_3} &= \frac{1}{2^N N!^2} \sum_{\sigma \in S_{2N}} (-1)^\sigma \int \frac{d^{2N} x}{(2\pi)^{2N}} \frac{d^{2N} y}{(2\pi)^{2N}} \prod_{j=1}^N F^{(0)}(x_j) F'^{(0)}(x_{N+j}) F^{(1)}(y_j) F'^{(1)}(y_{N+j}) \\ &\quad \prod_{j \in \mathcal{K}(\sigma)} (-1)^{s(\sigma(j)+s(j)+1)} \left[\frac{1}{2 \cosh \frac{x_j - y_{R\sigma(j)}}{2}} \frac{1}{2 \sinh \frac{y_{R\sigma(j)} - y_{\sigma(j)}}{2}} \frac{1}{2 \cosh \frac{y_{\sigma(j)} - x_{\sigma^{-1}R\sigma(j)}}{2}} \right. \\ &\quad \left. + \frac{1}{2 \cosh \frac{x_j - y_{R\sigma(j)}}{2}} \frac{1}{2 \sinh \frac{y_{R\sigma(j)} - y_{\sigma(j)}}{2}} \frac{1}{2 \cosh \frac{y_{\sigma(j)} - x_{\sigma^{-1}R\sigma(j)}}{2}} \right] \frac{1}{2 \sinh \frac{x_{\sigma^{-1}R\sigma(j)} - x_{R\sigma^{-1}R\sigma(j)}}{2}} \\ &= \frac{1}{2^{2N} N!^2} \sum_{\sigma \in S_{2N}} (-1)^\sigma \int \frac{d^N x}{(2\pi)^N} \prod_{j \in \mathcal{K}(\sigma)} (-1)^{s(\sigma(j)+s(j))} \rho(x_j, x_{R\sigma^{-1}R\sigma(j)}) \\ &= \frac{1}{N!} \sum_{\sigma \in S_N} (-1)^\sigma \int \frac{d^N x}{(2\pi)^N} \prod_{j=1}^N \rho_{\hat{D}_3}^{(\pm)}(x_j, x_{\sigma(j)}), \end{aligned} \quad (\text{A.20})$$

where

$$\begin{aligned} \rho_{\hat{D}_3}(x_1, x_2) &= -2 \int \frac{dy dy' dx'}{(2\pi)^3} \frac{1}{2 \cosh \frac{x-y}{2}} \left(F^{(0)}(y) \frac{1}{2 \sinh \frac{y-y'}{2}} F'^{(0)}(y') + F'^{(0)}(y) \frac{1}{2 \sinh \frac{y-y'}{2}} F^{(0)}(y') \right) \\ &\quad \frac{1}{2 \cosh \frac{y'-x'}{2}} \left(F^{(1)}(x') \frac{1}{2 \sinh \frac{x'-x}{2}} F'^{(1)}(x) + F'^{(1)}(x') \frac{1}{2 \sinh \frac{x'-x}{2}} F^{(1)}(x) \right). \end{aligned} \quad (\text{A.21})$$

Hence corresponding operator $\hat{\rho}_{\hat{D}_3}$ is

$$\begin{aligned} \hat{\rho}_{\hat{D}_3} &= \left(F^{(0)}(\hat{Q}) \tanh \frac{\hat{P}}{2} F'^{(0)}(\hat{Q}) + F'^{(0)}(\hat{Q}) \tanh \frac{\hat{P}}{2} F^{(0)}(\hat{Q}) \right) \frac{1}{2 \cosh \frac{\hat{P}}{2}} \\ &\quad \left(F^{(1)}(\hat{Q}) \tanh \frac{\hat{P}}{2} F'^{(1)}(\hat{Q}) + F'^{(1)}(\hat{Q}) \tanh \frac{\hat{P}}{2} F^{(1)}(\hat{Q}) \right) \frac{1}{2 \cosh \frac{\hat{P}}{2}} \\ &= \lim_{L \rightarrow 1} \hat{\rho}_{\hat{D}_{L+2}}. \end{aligned}$$

A.2 $\hat{A}_n \rightarrow U(N) + adj$.

Suppose the \hat{A}_n quiver theories without CS terms, where only one of the $U(N)$ vector multiplets is coupled to one fundamental hyper multiplet. This theory is related to the $U(N)$ gauge theory with $\mathcal{N} = 4$ vector multiplet, one adjoint hyper multiplet and $n+1$ fundamental

hyper multiplets. We can easily show this for the partition functions [47, 3, 45]. To be self contained, here we repeat its derivation. The density matrix operator $\hat{\rho}$ of the \hat{A}_n theory is

$$\rho_{\hat{A}_n} = \frac{1}{2 \cosh \frac{\hat{Q}}{2}} \frac{1}{2 \cosh^{n+1} \frac{\hat{P}}{2}} \quad (\text{A.22})$$

By the canonical transformation $(Q, P) \rightarrow (Q, -P)$, we get

$$\rho_{\hat{A}_n} = \frac{1}{2 \cosh \frac{\hat{P}}{2}} \frac{1}{2 \cosh^{n+1} \frac{\hat{Q}}{2}}. \quad (\text{A.23})$$

This $\hat{\rho}$ gives the N_f matrix model (A.1) with $N_f = n + 1$.

A.3 $\hat{D}_n \rightarrow USp + A$

We also review the proof of the 3d mirror symmetry between the partition functions on S^3 of the \hat{D}_n quiver and $USp + A$ theories. The gauge group of the \hat{D}_n quiver theory consists of four $U(N)$ nodes and $(n - 3)$ $U(2N)$ nodes, where one of $U(N)$ nodes associates one fundamental hypermultiplet. The partition function of this theory is given by

$$\begin{aligned} Z_{\hat{D}_n} = & \frac{1}{N!^2 (2N!)^{n-3}} \int \frac{d^N \mu}{(2\pi)^N} \frac{d^N \mu'}{(2\pi)^N} \frac{d^N \nu}{(2\pi)^N} \frac{d^N \nu'}{(2\pi)^N} \frac{d^{2N} \lambda^{(1)}}{(2\pi)^{2N}} \cdots \frac{d^{2N} \lambda^{(n-3)}}{(2\pi)^{2N}} \\ & \frac{\prod_{i \neq j} 2 \sinh \frac{\mu_i - \mu_j}{2} \cdot 2 \sinh \frac{\mu'_i - \mu'_j}{2}}{\prod_j 2 \cosh \frac{\mu_j}{2} \prod_{i,j} 2 \cosh \frac{\mu_i - \lambda_j^{(1)}}{2} \cdot 2 \cosh \frac{\mu'_i - \lambda_j^{(1)}}{2}} \\ & \prod_{\alpha=1}^{n-4} \left[\frac{\prod_{I \neq J} 2 \sinh \frac{\lambda_I^{(\alpha)} - \lambda_J^{(\alpha)}}{2} \cdot 2 \sinh \frac{\lambda_I^{(\alpha+1)} - \lambda_J^{(\alpha+1)}}{2}}{\prod_{I,J} 2 \cosh \frac{\lambda_I^{(\alpha)} - \lambda_J^{(\alpha+1)}}{2}} \right] \frac{\prod_{i \neq j} 2 \sinh \frac{\nu_i - \nu_j}{2} \cdot 2 \sinh \frac{\nu'_i - \nu'_j}{2}}{\prod_{i,j} 2 \cosh \frac{\nu_i - \lambda_j^{(n-3)}}{2} \cdot 2 \cosh \frac{\nu'_i - \lambda_j^{(n-3)}}{2}}. \end{aligned} \quad (\text{A.24})$$

Corresponding $\hat{\rho}$ is

$$\hat{\rho}_{\hat{D}_n} = \left\{ \frac{1}{2 \cosh \frac{\hat{Q}}{2}}, \tanh \frac{\hat{P}}{2} \right\} \left(\frac{1}{2 \cosh^{n-2} \frac{\hat{P}}{2}} \right)^{n-2} \tanh \frac{\hat{P}}{2} \left(\frac{1}{2 \cosh^{n-2} \frac{\hat{P}}{2}} \right)^{n-2}. \quad (\text{A.25})$$

By using

$$\left\{ \frac{1}{\cosh \frac{\hat{Q}}{2}}, \tanh \frac{\hat{P}}{2} \right\} = \frac{2}{\cosh \frac{\hat{P}}{2}} \left(\sinh \frac{\hat{P}}{2} \frac{1}{\cosh \frac{\hat{Q}}{2}} \cosh \frac{\hat{P}}{2} + \cosh \frac{\hat{P}}{2} \frac{1}{\cosh \frac{\hat{Q}}{2}} \sinh \frac{\hat{P}}{2} \right) \frac{1}{\cosh \frac{\hat{P}}{2}},$$

we find

$$\hat{\rho}_{\hat{D}_n} = \frac{1}{2 \cosh \frac{\hat{P}}{2}} \left(2 \sinh \frac{\hat{P}}{2} \frac{1}{2 \cosh \frac{\hat{Q}}{2}} 2 \cosh \frac{\hat{P}}{2} + 2 \cosh \frac{\hat{P}}{2} \frac{1}{2 \cosh \frac{\hat{Q}}{2}} 2 \sinh \frac{\hat{P}}{2} \right)$$

$$\left(\frac{1}{2 \cosh^{n-2} \frac{\hat{P}}{2}}\right)^{n-1} \tanh \frac{\hat{P}}{2} \left(\frac{1}{2 \cosh^{n-2} \frac{\hat{P}}{2}}\right)^{n-2}. \quad (\text{A.26})$$

Then the similarity transformation

$$\hat{\rho}_{\hat{D}_n} \rightarrow 2 \cosh \frac{\hat{P}}{2} \cdot \hat{\rho} \cdot \frac{1}{2 \cosh \frac{\hat{P}}{2}}, \quad (\text{A.27})$$

leads us to

$$\hat{\rho}_{\hat{D}_n} = \left(2 \sinh \frac{\hat{P}}{2} \frac{1}{2 \cosh \frac{\hat{Q}}{2}} 2 \cosh \frac{\hat{P}}{2} + 2 \cosh \frac{\hat{P}}{2} \frac{1}{2 \cosh \frac{\hat{Q}}{2}} 2 \sinh \frac{\hat{P}}{2}\right) \frac{2 \sinh \frac{\hat{P}}{2}}{\left(2 \cosh \frac{\hat{P}}{2}\right)^{2n}}. \quad (\text{A.28})$$

By the canonical transformation

$$(P, Q) \rightarrow (Q, -P), \quad (\text{A.29})$$

we obtain

$$\hat{\rho}_{\hat{D}_n} = \frac{2 \sinh \frac{\hat{Q}}{2}}{\left(2 \cosh \frac{\hat{Q}}{2}\right)^{2n}} \left(2 \sinh \frac{\hat{Q}}{2} \frac{1}{2 \cosh \frac{\hat{P}}{2}} 2 \cosh \frac{\hat{Q}}{2} + 2 \cosh \frac{\hat{Q}}{2} \frac{1}{2 \cosh \frac{\hat{P}}{2}} 2 \sinh \frac{\hat{Q}}{2}\right). \quad (\text{A.30})$$

Indeed this gives the same partition function as the $USp + A$ theory with $N_f = n$ because $\hat{\rho}$ of the $USp + A$ theory:

$$\hat{\rho}_{USp+A}(\hat{Q}, \hat{P}) = \frac{2 \cosh \frac{\hat{Q}}{2}}{\left(4 \cosh^2 \frac{\hat{Q}}{2}\right)^{n-1}} \frac{1}{2 \cosh \frac{\hat{P}}{2}}, \quad (\text{A.31})$$

satisfies

$$\begin{aligned} \hat{\rho}_{\hat{D}_n} \frac{1 - \hat{R}}{2} &= \frac{2 \sinh \frac{\hat{Q}}{2}}{\left(2 \cosh \frac{\hat{Q}}{2}\right)^{2n}} \left(2 \sinh \frac{\hat{Q}}{2} \frac{1 - \hat{R}}{2 \cosh \frac{\hat{P}}{2}} 2 \cosh \frac{\hat{Q}}{2} + \frac{\cosh^2 \frac{\hat{Q}}{2}}{\sinh \frac{\hat{Q}}{2}} \frac{1 - \hat{R}}{2 \cosh \frac{\hat{P}}{2}} 2 \cosh \frac{\hat{Q}}{2}\right) \\ &= \frac{1}{2 \cosh \frac{\hat{Q}}{2}} \left(\hat{\rho}_{USp+A} \frac{1 - \hat{R}}{2}\right) 2 \cosh \frac{\hat{Q}}{2}. \end{aligned} \quad (\text{A.32})$$

Thus, combining the results in app. A.1, app. A.2 and app. A.3, we prove (1.12).

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